The reproducing kernel Hilbert space approach in nonparametric regression problems with correlated observations

Djihad Benelmadani, Karim Benhenni, Sana Louhichi

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D. BENELMADANI, K. BENHENNI and S. LOUHICHI

Laboratoire Jean Kuntzmann (CNRS 5224), Université Grenoble Alpes, France.

djihad.benelmadani@univ-grenoble-alpes.fr, karim.benhenni@univ-grenoble-alpes.fr,

sana.louhichi@univ-grenoble-alpes.fr.

Abstract: In this paper we investigate the problem of estimating the regression function in models with correlated observations. The data is obtained from several experimental units each of them forms a time series. We propose a new estimator based on the inverse of the autocovariance matrix of the observations, assumed known and invertible. Using the properties of the Reproducing Kernel Hilbert spaces, we give the asymptotic expressions of its bias and its variance. In addition, we give a theoretical comparison, by calculating the IMSE, between this new estimator and the classical one proposed by Gasser and Müller. Finally, we conduct a simulation study to investigate the performance of the proposed estimator and to compare it to the Gasser and Müller's estimator in a finite sample set.

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1 Introduction

One of the situations that statisticians encounter in their studies is estimating a whole function based on partial observations of this function. For instance, in pharmacokinetics one wishes to estimate the concentration-time of some injected medicine in the organism, based on the observations of the concentration from blood tests over a period of time. In statistical terms, one wants to estimate a function, say g, relating two random variables: the explanatory variable X and the response variable Y, without any parametric restrictions on the function g. The statistical model often used is the following: $Y_i = g(X_i) + \varepsilon_i$ where $(X_i, Y_i)_{1\leq i\leq n}$ are n independent replicates of (X, Y) and $\{\varepsilon_i, i = 1, \cdots, n\}$ are centered errors.

The most intensively treated model has been the one in which $(\varepsilon_i)_{1\leq i\leq n}$ are independent errors and $(X_i)_{1\leq i\leq n}$ are fixed within some domain. We mention the work of [6, 15, 23]among others. However, the independence of the observations is not always a realistic assumption. For instance, the growth curve models are usually used in the case of longitudinal data, where the same experimental unit is being observed on multiple points of time. As a real life example, we consider the observation of the height growth of children, it is clear that the heights observed on the same child will be correlated. The temperature observations measured along the day are also correlated. For this, we focus, in this paper, on the nonparametric kernel estimation problem where the observations are correlated.

In the current paper, we consider a situation where the data is generated from m experimental units each of them having n measurements of the response. For this data, we consider the so-called fixed design regression model with repeated measurements given by,

$$Y_j(t_i) = g(t_i) + \varepsilon_j(t_i) \quad \text{for } i = 1, \cdots, n \text{ and } j = 1, \cdots, m, \tag{1}$$

where $\{\varepsilon_j, j = 1, \cdots, m\}$ is a sequence of i.i.d. centered error processes with the same distribution as a process ε. It is safe to relax the correlation assumption between the experimental units since the latters are usually chosen randomly.

This model is usually used in the growth curve analysis and dose response problems, see for instance, the work of [4]. It has also been considered by [21] with $m = 1$. He supposed that the observations are asymptotically independent when the number of observations tends to infinity, i.e., $\text{Cov}(\varepsilon(s), \varepsilon(t)) = O(1/n)$ for $s \neq t$, which is not a realistic assumption, for instance, in the growth curve analysis.

The correlated observations case was considered by [18], who investigated the estimation of g in Model (1) where ε is a stationary error process. Using the kernel estimator proposed by Gasser and Müller, see [15], they proved the consistency in $\mathbb{L}^2$ space of this estimator, when the number of experimental units m tends to infinity, but not when n tends to infinity as it is the case of independent observations.

The Assumption of stationarity made on the observations is however restrictive. In the previous pharmacokinetics example for instance, it is clear that the concentration of the medicine will be high at the beginning then decreases with time. For this, we shall investigate the estimation of g in Model (1) where ε is not necessarily a stationary error process. This case was partially investigated by [9, 13], where the Gasser and Müller estimator was used.

In this paper, we propose a new estimator for the regression function g in Model (1). This estimator, which is a linear kernel estimator, is based on the inverse of the autocovariance matrix of the observations, that we assume known and invertible.

The proposed estimator was inspired by the work of [25, 26, 27] but in a different context than ours. They considered the parametric model: $Y(t) = \beta f(t) + \varepsilon(t)$ where β is an unknown real parameter and f is a known function belonging to the Reproducing Kernel Hilbert Space associated to the autocovariance function of the error process ε, denoted by RKHS(R). They, similarly to us, assumed that the autocovariance matrix is known and invertible. It is worth noting that the Reproducing Kernel Hilbert Spaces have been used in several domains, for instance, in Statistics by [25] and more recently by [12], in Mathematical Analysis in [28] and in Signal processing in [24].

We compare the proposed estimator to the classical Gasser and Müller's estimator, proving in particular that, the proposed estimator has an asymptotically smaller variance. However the Gasser and Müller's estimator doesn't require the knowledge of the autocovariance function.

This paper is organized as follows. In section 2, we construct our proposed estimator for the function g in Model (1) where ε is a centered, second order error process with a continuous autocovariance function R. It is constructed through a Kernel K and a bandwidth $h = h(n)$, more precisely, through the following function defined, for $x \in [0,1]$, by,

$$f_{x,h}(t) = \int_0^1 R(s,t)\varphi_{x,h}(t)\, ds \quad \text{where} \quad \varphi_{x,h}(t) = \frac{1}{h}K\Big(\frac{x-s}{h}\Big), \quad \text{for} \quad t \in [0,1]. \qquad (2)$$

We shall see that this function belongs to the RKHS(R). This allows us to use the properties of this space to control the variance of the proposed estimator. These properties were introduced by [22] to solve various problems in statistical inference on time series. We also give, in this section, the analytical expressions of this estimator for the generalised Wiener process and the Ornstein Uhlenbeck process, since the analytical expression of the inverse of the autocovariance matrix is known for this class of processes.

In Section 3, we derive the asymptotic results of this estimator. We give an asymptotic expression of the weights of this linear estimator, which is used to derive the asymptotic expression of its bias. The properties of the RKHS(R) not only allow us to obtain the asymptotic expression of the variance, but also to find the optimal rate of convergence of the residual variance. After obtaining the asymptotic expression of the Integrated Mean Squared Error (IMSE), we derive the asymptotic optimal bandwidth with respect to the IMSE. Moreover, we prove the asymptotic normality of the proposed estimator.

In Section 4, we give a theoretical comparison between the new estimator and the Gasser and Müller's estimator in terms of the asymptotic variance. Proving that the proposed estimator has, asymptotically, a smaller variance than that of Gasser and Müller. When ε is a Wiener process, we compare the two estimators in terms of the IMSE .

In Section 5, we conduct a simulation study in order to investigate the performance of the proposed estimator in a finite sample set, then we compare it with the Gasser and Müller's estimator for different values of the number of experimental units and different values of the sample size. Since the classical cross-validation criterion is shown to be inefficient in the presence of correlation (see for instance, [1, 11, 19, 20]), we use the bandwidth that minimizes the (non-asymptotic) IMSE. The results of this simulation study confirm our theoretical statements given in Section 3 and Section 4.

Section 6 is dedicated to the proofs of the theoretical results. Finally, Section 7 is devoted to an appendix about the RKHS(R) and some technical details.

2 Construction of the estimator using the RKHS approach

We consider Model (1) where g is the unknown regression function on $[0,1]$ and $\{\varepsilon_j(t), t \in [0,1]\}_j$ is a sequence of error processes.
We assume that $g \in C^2([0,1])$ and that $(\varepsilon_j)_j$ are i.i.d. processes with the same distribution as a centered second order process ε. We denote by R its autocovariance function, assumed to be known, continuous and forms a non singular matrix when restricted to $T \times T$ for any finite set $T \subset [0,1]$.

2.1 Projection estimator

In this section, we shall give the definition of the new proposed estimator for the regression function g in Model (1). This estimator (see Defenition 1 below) is constructed using the

function $f_{x,h}$ given by (2) for $x \in [0,1]$, $h \in]0,1[$ and K is a first order kernel[1] of support $[-1,1]$ belonging to C^1.
This function is well known in time series analysis and has been used by several authors. We mention, among others, the work of [5] and [25] for linear regression models with correlated errors. It is mainly used due to its belonging to the Reproducing Kernel Hilbert Space associated to the autocovariance function R (see Appendix 1 for more details). This space is spanned by the functions $R(\cdot, t_i)_{1\le i\le n}$ forming a closed subspace on which an orthogonal projection of the function $f_{x,h}$ is feasible. We shall call the estimator obtained by this approach, the projection estimator.

The proposed estimator, which is a kernel estimator, is linear in the observations $\overline{Y}(t_i)$ and is given by the following definition.

Definition 1 *The projection estimator of the regression function g in the model* (1) *based on the observations of $(t_i, Y_j(t_i))_{\substack{1\le i\le n\\ 1\le j\le m}}$ is given for any $x \in [0,1]$ by,*

$$\hat{g}_n^{pro}(x) = \sum_{i=1}^{n} m_{x,h}(t_i)\overline{Y}(t_i), \tag{3}$$

where $\overline{Y}(t_i) = \frac{1}{m}\sum_{j=1}^{m} Y_j(t_i)$ and the weights $(m_{x,h}(t_i))_{1\le i\le n}$ are being determined, letting $T_n = (t_i)_{1\le i\le n}$, by,

$$m'_{x,h_{|T_n}} = f_{x,h_{|T_n}}{}' R^{-1}_{|T_n}, \tag{4}$$

with $f_{x,h_{|T_n}} := (f_{x,h}(t_1), \dots, f_{x,h}(t_n))'$, $R_{|T_n} := (R(t_i,t_j))_{1\le i,j\le n}$, $R^{-1}_{|T_n}$ the inverse of $R_{|T_n}$ and $m_{x,h_{|T_n}} := (m_{x,h}(t_1), \dots, m_{x,h}(t_n))'$, where v' denotes the transpose of a vector v.

This estimator appears to be complicated and uneasy to compute due to its dependence on the inverse of the covariance matrix but the following propositions show that, for some classical error processes, such as the Wiener and the Ornstein-Uhlenbeck processes, it has a simplified expression.

Proposition 1 *Consider the regression model* (1) *where ε is of autocovariance function $R(s,t) = \int_0^{\min(s,t)} u^\beta\, du$ for a positive constant β. Let $t_0 = 0$, $t_{n+1} = 1$ and set $\overline{Y}(t_0) = 0$ and $\overline{Y}(t_{n+1}) = \overline{Y}(t_n)$. For any $x \in [0,1]$, the projection estimator* (3) *equals to,*

$$\begin{aligned}\hat{g}_n^{pro}(x) = \frac{1}{\beta+1}\Bigg(&\sum_{i=1}^{n+1}\overline{Y}(t_i)\int_{t_{i-1}}^{t_i}\varphi_{x,h}(s)ds \\ &+ \sum_{i=0}^{n-1}\frac{\overline{Y}(t_{i+1})-\overline{Y}(t_i)}{t_{i+1}^{\beta+1}-t_i^{\beta+1}}\int_{t_i}^{t_{i+1}}(s^{\beta+1}-t_{i+1}^{\beta+1})\varphi_{x,h}(s)ds\Bigg),\end{aligned} \tag{5}$$

Remark 1 *Taking $\beta = 0$ in the previous proposition gives the expression of the projection estimator* (3) *in the case where ε is the classical standard Wiener error process.*

[1]The kernel K satisfies: $\int_{-1}^{1} K(t)dt = 1$, $\int_{-1}^{1} tK(t)dt = 0$ and $\int_{-1}^{1} t^2K(t)dt < +\infty$.

Proposition 2 *If the error process ε in Model* (1) *is the Ornstein-Uhlenbeck process with $R(s,t) = e^{-|t-s|}$ then for any $x \in [0,1]$,*

$$\hat{g}_n^{pro}(x) = \sum_{i=2}^{n-1} \overline{Y}(t_i) \int_{t_{i-1}}^{t_{i+1}} e^{|s-t_i|} \varphi_{x,h}(s)\, ds + \overline{Y}(t_1) \int_0^{t_2} e^{s-t_1} \varphi_{x,h}(s)\, ds$$

$$+ \overline{Y}(t_n) \int_{t_{n-1}}^{1} e^{t_n - s} \varphi_{x,h}(s)\, ds - \sum_{i=1}^{n-1} \frac{e^{t_{i+1}} \overline{Y}(t_{i+1}) - e^{t_i} \overline{Y}(t_i)}{1 - e^{-2(t_{i+1}-t_i)}} \int_{t_i}^{t_{i+1}} e^{-s} \varphi_{x,h}(s)\, ds$$

$$+ \sum_{i=1}^{n-1} \frac{e^{-t_{i+1}} \overline{Y}(t_{i+1}) - e^{-t_i} \overline{Y}(t_i)}{1 - e^{-2(t_{i+1}-t_i)}} \int_{t_i}^{t_{i+1}} e^{s} \varphi_{x,h}(s)\, ds,$$

where $\varphi_{x,h}$ is defined in the previous proposition.

Remark 2 *As the previous propositions show, the expression of $m_{x,h_{|T_n}}$ is known analytically for error processes of practical interest. For more complicated error processes, numerical methods can be used as we shall see in the simulation study section. For more general error processes, we will give an asymptotic expression of the weights of the projection estimator (see Lemma 3 below).*

2.2 Assumptions and comments

In order to derive our asymptotic results, the following assumptions on the autocovariance function R and the Kernel K are required.

(A) R is continuous on the entire unit square and has left and right derivatives up to order two at the diagonal (when $s = t$), i.e.,

$$R^{(0,1)}(t, t^-) = \lim_{s \uparrow t} \frac{\partial R(t,s)}{\partial s} \quad \text{and} \quad R^{(0,1)}(t, t^+) = \lim_{s \downarrow t} \frac{\partial R(t,s)}{\partial s},$$

exist and are continuous (in a similar way we define $R^{(0,2)}(t,t^-)$ and $R^{(0,2)}(t,t^+)$). Off the diagonal (when $s \neq t$ in the unit square), R has continuous derivatives up to order two.

For $t \in]0,1[$, let $\alpha(t) = R^{(0,1)}(t,t^-) - R^{(0,1)}(t,t^+)$. Assumption (A) gives the following lemma concerning the jump function α.

Lemma 1 *If Assumption (A) is satisfied then the jump function α is a positive function.*

To obtain our asymptotic results, we shall give next a stronger assumption on the jump function α.

(B) We assume that α is Lipschitz on $]0,1[$, that $\inf_{0<t<1} \alpha(t) = \alpha_0 > 0$ and that $\sup_{0<t<1} \alpha(t) = \alpha_1 < \infty$.

Assumptions (A) and (B) are classical regularity conditions and were used in several works (see for instance [5, 25, 29]).

(C) For each $t \in [0,1]$, $R^{(0,2)}(.,t^+)$ is in the Reproducing Kernel Hilbert space associated to R, denoted by RKHS(R), equipped with the norm $|| \cdot ||$. In addition, $\sup_{0 \le t \le 1} ||R^{(0,2)}(.,t^+)|| < \infty$ (see Appendix for more details).

Assumption (C), which is more restrictive than (B) as indicated by [25], is necessary to evaluate the weights of the projection estimator (see Lemma 3 below).

(D) K is an even function and K' is a Lipschitz function on $[-1, 1]$.

Examples of autocovariance functions which satisfy Assumptions (A), (B) and (C) are given as follows.

Example 1

1. *The autocovariance function $R(s,t) = \sigma^2 min(s,t)$ of the Wiener process, has a constant jump function $\alpha(t) = \sigma^2$ and $R^{(i,j)}(s,t) = 0$ for all integers i, j such that $i + j = 2$ and $s \neq t$.*

2. *The autocovariance function $R(s,t) = \sigma^2 e^{-\lambda|s-t|}$ of the stationary Ornstein-Uhnelbeck process with $\sigma > 0$ and $\lambda > 0$. For this process the jump function is $\alpha(t) = 2\sigma^2\lambda$ and $R^{(0,2)}(s,t) = \sigma^2\lambda^2 e^{-\lambda|s-t|}$.*

3. *Another general class of autocovariance functions was given by [25] and has the form,*
$$R(s,t) = \int_0^{1/|t-s|} (1 - \mu|t-s|)p(\mu)\, d\mu,$$
where p is a probability density and p' its derivative are such that,
$$\lim_{\mu\to\infty} \mu^3 p(\mu) < \infty, \quad \text{and} \quad \int_a^{\infty} (\mu p'(\mu) + 3p(\mu))^2)\mu^6 d\mu < \infty,$$
for some a. We have $\alpha(t) = 2\int_0^\infty up(u)\, du$.

3 Local asymptotic results

Let $T_n = (t_{i,n})_{1\le i\le n}$ for $n \ge 1$, be a fixed sequence of designs with $T_n \in D_n$, where,
$$D_n = \{(s_1, s_2, \dots, s_n) \ :\ 0 \le s_1 < s_2 < \cdots < s_n \le 1\}.$$
Set $t_{0,n} = 0, t_{n+1,n} = 1$, $d_{j,n} = t_{j+1,n} - t_{j,n}$ and let for $x \in [0, 1]$, $h = h(n)$,
$$I_{x,h} = \{i = 1, \cdots, n \ :\ [t_{i-1,n}, t_{i+1,n}] \cap]x-h, x+h[\neq \emptyset\}.$$
Denote by $N_{T_n} = \text{Card}(I_{x,h})$. Recall that $[x-h, x+h]$ is the support of the function $\varphi_{x,h}$. To obtain the asymptotic results, we require that the sequence $(T_n)_{n\ge1}$ satisfies the next assumption.

(E) $\lim\limits_{n\to\infty} \sup\limits_{0\le j\le n} d_{j,n} = 0$, $\lim\limits_{n\to\infty} \frac{1}{h} \sup\limits_{0\le j\le n} d_{j,n} = 0$, $\lim\limits_{n\to\infty} N_{T_n} \frac{1}{h^2} \sup\limits_{0\le j\le n} d_{j,n}^2 = 0$ and $\limsup\limits_{n\to\infty} (N_{T_n}^2 \frac{1}{h^2} \sup\limits_{0\le j\le n} d_{j,n}^2) < \infty$.

A simple sequence of designs that verifies Assumption (E) was presented by [27] as follows.

Definition 2 *Let F be a distribution function of some density function f such that $\sup\limits_{0<t<1} f(t) < \infty$ and $\inf\limits_{0<t<1} f(t) > 0$. The so-called regular sequence of designs generated by f is defined by,*
$$T_n = \left\{ t_{i,n} = F^{-1}\left(\frac{i}{n}\right), i = 1, \dots, n \right\}.$$

In the sequel, the density f is assumed to be at least in $C^2([0,1])$. This sequence of designs verifies the following Lemma (see for instance Benelmadani et al. (2018a) for a proof).

Lemma 2 *Let $(T_n)_{n\geq 1}$ be a regular sequence of designs generated by some density function. For $x \in]0,1[$ and $h>0$, suppose that $T_n \cap [x-h, x+h] \neq \emptyset$ and that $nh \geq 1$. Then,*

$$\sup_{0\leq j\leq n} d_{j,n} = O\Big(\frac{1}{n}\Big) \quad \text{and} \quad N_{T_n} = O(nh), \tag{6}$$

where N_{T_n} and $d_{j,n}$ are defined as above. In addition, if $\lim\limits_{n\to\infty} nh = \infty$ then the regular sequence verifies Assumption (E).

3.1 Evaluation of the bias

In order to derive the asymptotic expression of the bias term of the projection estimator, we shall first give the asymptotic approximation of the weights $m_{x,h|T_n}$ (defined by (4)) in the following lemma.

Lemma 3 *Suppose that Assumptions $(A), (B)$ and (C) are satisfied. Then for any $x \in]0,1[$,*

$$m_{x,h}(t_{i,n}) = \begin{cases} \frac{1}{2}\varphi_{x,h}(t_{i,n})(t_{i+1,n} - t_{i-1,n}) + O\big(\alpha_{n,h} + \beta_{n,h}\big) & \text{if } i \notin \{1,n\} \text{ and} \\ & [t_{i-1,n}, t_{i+1,n}] \cap [x-h, x+h] \neq \emptyset, \\ O\big(N_{T_n}\alpha_{n,h} + n\beta_{n,h}\big) & \text{if } i \in \{1,n\}, \\ O\big(\beta_{n,h}\big) & \text{otherwise,} \end{cases}$$

where,

$$\alpha_{n,h} = \sup_{0\leq i\leq n} \sup_{t_{i,n}\leq s,t\leq t_{i+1,n}} d_{i,n} \, |\alpha(s)\varphi_{x,h}(s) - \alpha(t)\varphi_{x,h}(t)| = O\Big(\frac{1}{h^2}\sup_{0\leq j\leq n} d_j^2\Big),$$

$$\beta_{n,h} = \sup_{0\leq t\leq 1} \frac{1}{2\alpha(t)} ||R^{(0,2)}(.,t)|| \frac{\sqrt{C}}{\sqrt{h}} \sup_{0\leq j\leq n} d_j^2 = O\Big(\frac{1}{\sqrt{h}}\sup_{0\leq j\leq n} d_j^2\Big),$$

and C is a positive constant defined in Proposition 5 below. Recall that $A_n = O(n)$ means that $\overline{\lim\limits_{n\to\infty}} \frac{|A_n|}{n} < +\infty$.

Remark 3 *This Lemma allows to see that the weights of the projection estimator are asymptoticly equivalent to those of some well known linear estimators of the regression function g. For instance,*

- *Priestly and Chao (see [6, 23]) used the following weights:*

$$W_{x,h}(t_i) = (t_{i+1,n} - t_{i,n})\varphi_{x,h}(t_i) \quad \text{for } i = 1, \cdots, n.$$

- *Gasser and Müller (see [15]) used the following weights:*

$$W_{x,h}(t_i) = \int_{s_{i-1,n}}^{s_{i,n}} \varphi_{x,h}(s)\, ds \quad \text{for } i = 1, \cdots, n,$$

where, $s_0 = 0$, $s_n = 1$ and $s_{i,n} = (t_{i+1,n} - t_{i,n})$ for $i = 1, \cdots, n-1$.

- *Cheng and Lin (see [10]) replaced $s_{i,n}$ by $t_{i,n}$, in the weights of the Gasser and Müller estimator.*

Using the asymptotic approximation of the weights given in Lemma 3, we can obtain the asymptotic expression of the bias of the projection estimator as shows the following proposition.

Proposition 3 *Suppose that Assumptions $(A)-(D)$ are satisfied. If $T_n \cap]x-h, x+h[\neq \emptyset$ and $nh \geq 1$, then for any $x \in]0,1[$,*

$$\mathbb{E}(\hat{g}_n^{pro}(x)) - g(x) = \frac{1}{2}h^2 g''(x)B + o(h^2) + O\Big(\frac{N_{T_n}}{h^3} \sup_{0\leq j\leq 1} d_{j,n}^3 + N_{T_n}\alpha_{n,h} + n\beta_{n,h}\Big),$$

where $\alpha_{n,h}$ and $\beta_{n,h}$ are given in Lemma 3 and $B = \int_{-1}^{1} t^2 K(t)\ dt$.

Remark 4 *Under the assumption of Lemma 2 we have,*

$$\mathbb{E}\ (\hat{g}_n^{pro}(x)) - g(x) = \frac{1}{2}h^2 g''(x)B + o(h^2) + O\Big(\frac{1}{nh}\Big).$$

When ε in Model (1) is a Wiener process, a direct computation of the bias term of the projection estimator (5), with $\beta = 0$, show that the term $O\Big(\frac{1}{nh}\Big)$ can be improved. The result is given by the following proposition.

Proposition 4 *Consider Model (1) with a Wiener error process of autocovariance function $R(s,t) = \min(s,t)$. Let $(T_n)_{n\geq 1}$ be a regular sequence of designs generated by a density function f (cf Definition 2) and let K be a kernel satisfying Assumption (D). If $T_n \cap]x-h, x+h[\neq \emptyset$ and $nh \geq 1$ then,*

$$\mathbb{E}\ (\hat{g}_n^{pro}(x)) - g(x) = \frac{1}{2}h^2 g''(x)B + o(h^2) + O\Big(\frac{1}{n^2 h}\Big),$$

where B is given in Proposition 3 above.

3.2 Evaluation of the variance

It is shown in Lemma 5 of the Appendix that $f_{x,h}$ defined by (2) belongs to the RKHS(R) equipped with its norm $||\ ||$. Furthermore, by Lemma 5 of Appendix we have,

$$||f_{x,h}||^2 = \int_0^1 \int_0^1 \varphi_{x,h}(s) R(s,t) \varphi_{x,h}(t) ds\ dt \triangleq \sigma_{x,h}^2. \tag{7}$$

In addition if $P_{|T_n} f_{x,h}$ is the projection of $f_{x,h}$ on the subspace of $\mathcal{F}$ spanned by $\{R(.,t), t \in T_n\}$ then it is shown by (F2) in the supplement fact of the Appendix that,

$$||P_{|T_n} f_{x,h}||^2 = m\mathrm{Var}\ \hat{g}_n^{pro}(x), \tag{8}$$

The following proposition controls the residual variance $\frac{\sigma_{x,h}^2}{m} - \mathrm{Var}\ \hat{g}_n^{pro}(x)$.

Proposition 5 *Suppose that Assumptions (A) and (B) are satisfied. Moreover, assume that $\frac{1}{h}\sup_{1\le i\le n} d_i \le 1$ and let,*

$$K_\infty = \sup_{t\in[-1,1]} |K(t)|, \quad R_1 = \sup_{t,s\in[0,1]} |R^{(1,1)}(s-,t+)| \quad \textit{and} \quad R_2 = \sup_{t,s\in[0,1]} |R^{(0,2)}(s,t+)|.$$

Then we have for any $x \in]0,1[$,

$$0 \le \frac{\sigma^2_{x,h}}{m} - \mathrm{Var}\ \hat{g}_n^{pro}(x) \le \frac{C}{mh} \sup_{0\le j\le n} d^2_{j,n},$$

where $C = \begin{cases} K^2_\infty(\frac{4}{3}\alpha_1 + R_1 + \frac{4}{3}R_2) & \textit{if } (x-h) \textit{ and } (x+h) \in T_n, \\ K^2_\infty(\frac{8}{3}\alpha_1 + \frac{5}{3}R_1 + \frac{8}{3}R_2) & \textit{otherwise.} \end{cases}$

If moreover $\{T_n, n \ge 1\}$ satisfies Assumption (E) then Proposition 5 gives,

$$\lim_{n,m\to\infty} \Big(\mathrm{Var}\ \hat{g}_n^{pro}(x) - \frac{\sigma^2_{x,h}}{m}\Big) = 0.$$

The next proposition gives the rate of convergence of this residual variance.

Proposition 6 *Suppose that Assumptions $(A), (B)$ and (C) are satisfied. Moreover, assume that $(T_n)_{n\ge1}$ is a sequence of designs verifying Assumption (E). Then for any $x \in]0,1[$ and for any positive integer m,*

$$\varliminf_{n\to\infty} \frac{mN^2_{T_n}}{h}\Big(\frac{\sigma^2_{x,h}}{m} - \mathrm{Var}\ \hat{g}_n^{pro}(x)\Big) \ge \frac{1}{12}\alpha(x)\Big\{\int_{-1}^1 K^{2/3}(t)dt\Big\}^3, \tag{9}$$

where $\sigma^2_{x,h}$ is given by (7).

Using Propositions 5 and 6 we can obtain the optimal convergence rate $1/(mn^2h)$, in the sense given by [17], of the residual variance. The result is given in the following proposition.

Proposition 7 *Suppose that all the assumptions of Lemma 2, Propositions 5 and 6 are satisfied. Then there exist some positive constants C and C' such that for any $x \in]0,1[$ and for any positive integer m,*

$$\varlimsup_{n\to\infty} mn^2h\Big(\frac{\sigma^2_{x,h}}{m} - \mathrm{Var}(\hat{g}_n^{pro}(x))\Big) \le C, \tag{10}$$

and,

$$\varliminf_{n\to\infty} mn^2h\Big(\frac{\sigma^2_{x,h}}{m} - \mathrm{Var}\ \hat{g}_n^{pro}(x)\Big) \ge C'. \tag{11}$$

For stronger assumptions on the kernel K and the on design T_n (instead of Assumption (E) we take a regular sequence of designs as introduced by Definition 2), we obtain the asymptotic expression of the variance as shows the following proposition.

Proposition 8 *Suppose that Assumptions $(A)-(D)$ are satisfied. Moreover assume that $(T_n)_{n\geq 1}$ is a regular sequence of designs generated by a density function f (see Definition 2). If $\lim_{n\to\infty} h = 0$ and $\lim_{n\to\infty} nh = \infty$ then for any $x \in]0,1[$,*

$$\mathrm{Var}(\hat{g}_n^{pro}(x)) = \frac{\sigma_{x,h}^2}{m} - \frac{1}{12mn^2}\int_{x-h}^{x+h}\frac{\alpha(t)}{f^2(t)}\varphi_{x,h}^2(t)dt + O\Big(\frac{1}{mn^3h^2}\Big), \tag{12}$$

where $\sigma_{x,h}^2$ is given by (7).

The following lemma, see [9], gives the expression of the main term of the asymptotic variance $\sigma_{x,h}^2/m$ in terms of h.

Lemma 4 *Suppose that Assumptions (A), (B) and (D) are satisfied. Then for any $x \in]0,1[$, $\sigma_{x,h}^2$ (as given by (7)) can be written as follows,*

$$\sigma_{x,h}^2 = \big(R(x,x) - \frac{1}{2}\alpha(x)C_K h\big) + o(h), \tag{13}$$

where $C_K = \int_{-1}^{1}\int_{-1}^{1}|u-v|K(u)K(v)dudv$.

3.3 IMSE and optimal bandwidth

Proposition 8 and Remark 4 allow to derive the asymptotic expression of the Mean Squared Error (MSE) and the Integrated Mean Squared Error (IMSE) of the projection estimator (3) given, without proof, in the next theorem.

Theorem 1 *If all the assumptions of Propositions 3 and 8 are satisfied and if $(T_n)_{n\geq 1}$ is a regular sequence of designs generated by some density function (see Definition 2) then for any $x \in]0,1[$,*

$$\begin{aligned}\mathrm{MSE}(\hat{g}_n^{pro}(x)) = {} & \frac{1}{m}\Big(R(x,x) - \frac{1}{2}\alpha(x)C_K h\Big) + \frac{1}{4}h^4(g''(x))^2B^2 + o\Big(h^4 + \frac{h}{m}\Big)\\ & + O\Big(\frac{1}{mn^2h} + \frac{h}{n} + \frac{1}{n^2h^2}\Big),\end{aligned}$$

$$\begin{aligned}\mathrm{IMSE}(\hat{g}_n^{pro}) = {} & \frac{1}{m}\int_0^1 R(x,x)w(x)\,dx - \frac{C_K h}{2m}\int_0^1 \alpha(x)w(x)\,dx\\ & + \frac{B^2}{4}h^4\int_0^1 [g''(x)]^2 w(x)\,dx + o\Big(h^4 + \frac{h}{m}\Big) + O\Big(\frac{1}{mn^2h} + \frac{h}{n} + \frac{1}{n^2h^2}\Big),\end{aligned}$$

where w is a positive density function, B and C_K are given in Propositions 3 and 8.

Remark 5 *We note here that the term $\frac{1}{12mn^2}\int_{x-h}^{x+h}\frac{\alpha(t)}{f^2(t)}\varphi_{x,h}^2(t)dt$ appearing in the asymptotic variance, does not appear in the asymptotic* MSE *and* IMSE*, because it is negligible comparing to the squared bias. However in the case of a Wiener error process (see Proposition 4), we get the term $O(\frac{1}{n^2h})$ instead of the term $O(\frac{1}{nh})$ in the asymptotic bias, which improves the rates in the expressions of the asymptotic MSE and IMSE.*

When ε is a Wiener process, the MSE *and* IMSE *of the projection estimator* (5) *(with $\beta = 0$) are given in Theorem 2 below.*

Theorem 2 *Consider Model* (1) *with a Wiener error process and suppose that the kernel K verifies Assumption (D). Moreover, assume that $(T_n)_{n\geq 1}$ is a regular sequence of designs generated by a function f (see Definition 2). If $\lim_{n\to\infty} h = 0$ and $\lim_{n\to\infty} nh = \infty$ then for any $x \in]0,1[$,*

$$\mathrm{MSE}(\hat{g}_n^{pro}(x)) = \frac{1}{m}\Big(R(x,x) - \frac{1}{2}\alpha(x)C_K h\Big) - \frac{1}{mn^2h}\frac{\alpha(x)}{f^2(x)}\int_{-1}^{1} K^2(t)\, dt$$
$$+ \frac{1}{4}h^4[g''(x)]^2B^2 + o\Big(\frac{h}{m} + h^4\Big) + O\Big(\frac{h}{n^2} + \frac{1}{mn^3h^2} + \frac{1}{mn^2} + \frac{1}{n^4h^2}\Big),$$

and the IMSE *is given by,*

$$\mathrm{IMSE}(\hat{g}_n^{pro}) = \frac{1}{m}\int_0^1 R(x,x)w(x)\, dx - \frac{C_K h}{2m}\int_0^1 \alpha(x)w(x)\, dx$$
$$- \frac{A}{12mn^2h}\int_0^1 \frac{\alpha(x)}{f^2(x)}w(x)\, dx + \frac{B^2}{4}h^4\int_0^1 [g''(x)]^2 w(x)\, dx + o\Big(\frac{h}{m} + h^4\Big)$$
$$+ O\Big(\frac{h}{n^2} + \frac{1}{mn^3h^2} + \frac{1}{mn^2} + \frac{1}{n^4h^2}\Big),$$

where w is a positive density function, $A = \int_{-1}^{1} K^2(t)\, dt$, B and C_K are given in Propositions 3 and 8.

The asymptotic optimal bandwidth is obtained by minimizing the asymptotic IMSE and is given in the following corollary.

Corollary 1 (Optimal bandwidth) *Suppose that the assumptions of Theorem 1 are satisfied. Moreover assume that $\frac{n}{m} = O(1)$ as $n, m \to \infty$. Denote by IMSE(h) the IMSE of the projection estimator when the bandwidth h is used. Then the bandwidth,*

$$h^* = \left(\frac{C_K \int_0^1 \alpha(x)w(x)\, dx}{2B\int_0^1 [g''(x)]^2 w(x)\, dx}\right)^{1/3} m^{-1/3}, \tag{14}$$

with B and C_K given in Propositions 3 and 8, is optimal in the sense that,

$$\overline{\lim_{n,m\to\infty}} \frac{\mathrm{IMSE}(h^*)}{\mathrm{IMSE}(h_{n,m})} \leq 1,$$

for any sequence of bandwidths $h_{n,m}$ verifying:

$$\lim_{n,m\to\infty} h_{n,m} = 0 \quad \text{and} \quad \overline{\lim_{n,m\to\infty}} m h_{n,m}^3 < +\infty.$$

3.4 Asymptotic normality

The next theorem presents the asymptotic normality of the projection estimator (3) for any error process ε.

Theorem 3 *Suppose that the assumptions of Theorem 1 are satisfied. Moreover assume that $\frac{n}{m} = O(1)$ as $n, m \to \infty$, that $\lim_{n,m\to\infty} nh^2 = \infty$ and that $\lim_{n,m\to\infty} \sqrt{m}h^2 = 0$. Then for any $x \in]0,1[$,*

$$\sqrt{m}\Big(\hat{g}_n^{pro}(x) - g(x)\Big) \xrightarrow{\mathscr{D}} Z \quad \text{with } Z \sim \mathcal{N}(0, R(x,x)) \quad \text{as } n, m \to \infty,$$

where $\mathscr{D}$ denotes the convergence in distribution and $\mathcal{N}$ is the normal distribution.

4 Comparison with the Gasser and Müller's estimator

In this section, we shall perform a theoretical comparison between the projection estimator given in (3) and the classical estimator proposed by Gasser and Müller (see [15]) that we recall in definition below.

Definition 3 *The Gasser and Müller's estimator of the regression function g based on the observations $(t_i, Y_j(t_i))_{\substack{1\le i\le n\\ 1\le j\le m}}$ is given for any $x \in [0,1]$ by,*

$$\hat{g}_n^{GM}(x) = \sum_{i=1}^{n} \overline{Y}(t_i) \int_{s_{i-1}}^{s_i} \varphi_{x,h}(s)\, ds\ , \tag{15}$$

where $\overline{Y}, \varphi_{x,h}$ and h are given in Definition 1. The midpoints $(s_i)_{1\le i\le n}$ are such that: $s_0 = 0, s_n = 1$ and for $i = 1, \ldots, n-1$, $s_i = (t_i + t_{i+1})/2$.

In order to compare this estimator to the projection estimator with respect to the IMSE, we recall in the next theorem the asymptotic expression of the IMSE of the Gasser and Müller's estimator (for the proof see [8, 9] for further detailed results).

Theorem 4 *Suppose that Assumptions (A), (B) and (D) are satisfied. Moreover assume that $(T_n)_{n\ge1}$ is a regular sequence of designs generated by a density function f (see Definition 2). If $\lim_{n\to\infty} h = 0$ and $\lim_{n\to\infty} nh = \infty$ then for any $x \in]0,1[$,*

$$\begin{aligned}\mathrm{MSE}(\hat{g}_n^{GM}(x)) = {} & \frac{1}{m}\Big(R(x,x) - \frac{1}{2}\alpha(x) C_K h\Big) + \frac{1}{4}h^4 (g''(x))^2 B^2 + o\Big(h^4 + \frac{h}{m}\Big) \\ & + O\Big(\frac{h}{n^2} + \frac{1}{n^4h^2} + \frac{1}{mn^3h^2} + \frac{1}{mn^2}\Big).\end{aligned}$$

The IMSE *is given by,*

$$\begin{aligned}\mathrm{IMSE}(\hat{g}_n^{GM}) = {} & \frac{1}{m}\int_0^1 R(x,x) w(x)\, dx - \frac{C_K h}{2m}\int_0^1 \alpha(x) w(x)\, dx + \frac{B^2}{4}h^4 \int_0^1 [g''(x)]^2 w(x)\, dx \\ & + o\Big(h^4 + \frac{h}{m}\Big) + O\Big(\frac{h}{n^2} + \frac{1}{n^4h^2} + \frac{1}{mn^3h^2} + \frac{1}{mn^2}\Big),\end{aligned}$$

where B and C_K are given in Propositions 3 and 8 and w is a continuous positive density.

The following theorem gives an asymptotic comparison in term of the variance of the projection estimator (3) and the Gasser and Müller's estimator (15).

Theorem 5 *If the assumptions of Proposition 8 are satisfied then for any $x \in]0,1[$,*

$$\lim_{n,m\to\infty} mn^2 h\ \Big(\mathrm{Var}\ \hat{g}_n^{GM}(x) - \mathrm{Var}\ \hat{g}_n^{pro}(x)\Big) = \frac{1}{12}\frac{\alpha(x)}{f^2(x)} > 0.$$

For a comparison of the bias of these estimators, we mention that the Gasser and Müller's estimator converges to zero slightly faster than the bias of the projection estimator, i.e., the term $O(\frac{1}{nh})$ in the bias of the projection estimator (see Remark 4) is replaced by $O(\frac{1}{n^2h})$ in the bias of the Gasser and Müller's estimator (see [8]). However, for the Wiener error process both estimators have the same bias convergence rates, thus we can compare the asymptotic IMSE of both estimators in the following theorem.

Theorem 6 *Consider Model* (1) *where ε is a Wiener error process. Suppose that the assumptions of Theorem 2 are satisfied. Moreover, assume that* $\lim_{n\to\infty} nh^2 = 0$ *and that* $\frac{m}{n} = O(1)$ *then,*

$$\lim_{n,m\to\infty} mn^2h \left(\text{IMSE}\left(\hat{g}_n^{GM}\right) - \text{IMSE}\left(\hat{g}_n^{pro}\right)\right) = \frac{1}{12}\int_0^1 \frac{\alpha(x)}{f^2(x)} w(x)\, dx > 0.$$

Remark 6 *Theorems 5 and 6 show that, the projection estimator has an asymptotically smaller variance than the Gasser and Müller's estimator for any error process, it also has an asymptotically smaller IMSE when ε is a Wiener error process. However the Gasser and Müller's estimator doesn't require the knowledge of the autocovariance function whereas the projection estimator does.*

5 Simulation study

In this section, we investigate the performance of the proposed estimator (3) using finite values of experimental units m and sampling points n. The following growth curves are considered:

$$\text{(M1)} \qquad g(x) = 10x^3 - 15x^4 + 6x^5 \quad \text{for} \quad 0 < x < 1.$$

$$\text{(M2)} \qquad g(x) = x + 0.5\, e^{-80(x-0.5)^2} \quad \text{for} \quad 0 < x < 1.$$

This growth curve was used by [9, 18], due to its similarity in shape to that of the logistic function, which is frequently found in growth curve analysis as noted by [18]. The sampling points are taken to be:

$$t_i = (i - 0.5)/n \quad \text{for } i = 1, \cdots, n. \tag{16}$$

The error process ε is taken to be the Wiener error process with autocovariance function $R(s,t) = \sigma^2 \min(s,t)$. The Kernel used here is the Epanechnikov kernel given by $K(u) = (3/4)(1 - u^2)I_{[-1,1]}(u)$ and the bandwidth is the optimal one with respect to the exact IMSE. We consider the mean of all estimators obtained from 100 simulations. We take $\sigma^2 = 0.5$. Simulations for other values of σ^2 gave similar results. The results are given in Figures 1 and 2 for a fixed number of observations $n = 100$ and three different values of experimental units $m = 5, 20, 50$.

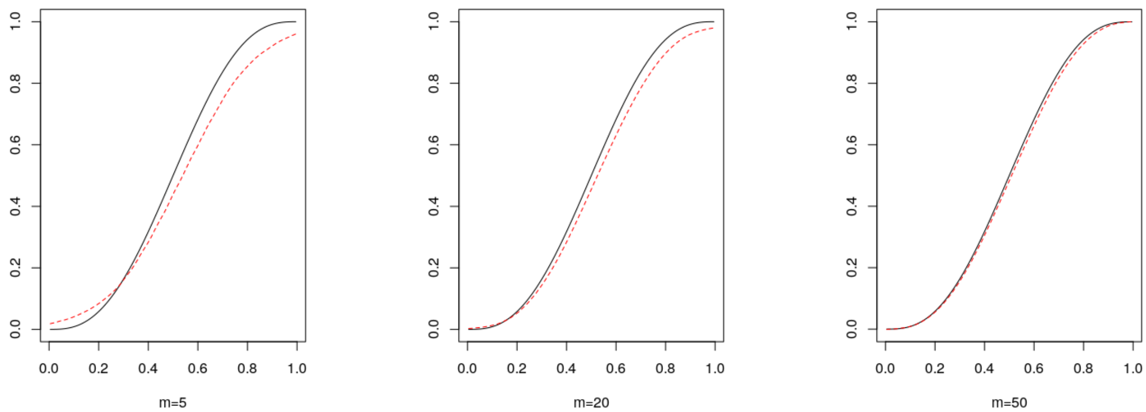

Figure 1: The regression function of model (M1) is in plain line and the projection estimator is in dashed line.

We can see, from Figure 1, that the projection estimator gets closer to the regression function when m gets bigger, which proves its good performance and consistency when m increases. These results are confirmed for another growth curve given in the following figure.

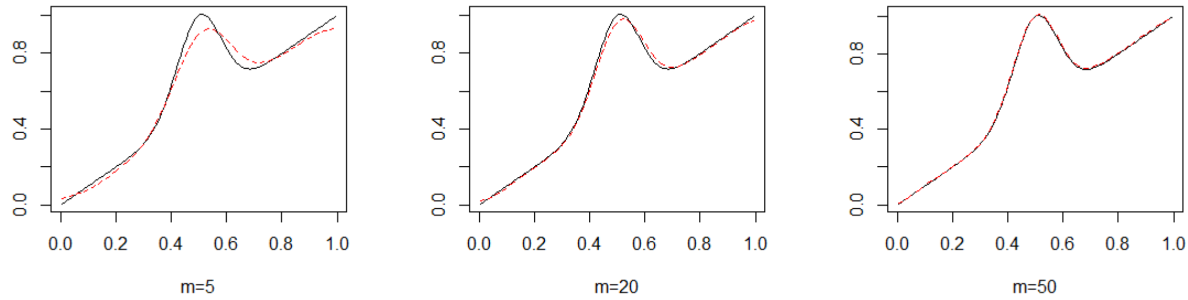

Figure 2: The regression function of model (M2) is in plain line and the projection estimator is in dashed line.

In this simulation study, we consider the comparison of the proposed estimator (3) to the Gasser and Müller (15) (referred here by GM estimator) with respect to the non-asymptotic IMSE in finite sample set. For this, we consider again the cubic growth curve of model (M1). We consider also the uniform design given by (16), the quartic kernel $K(u) = (15/16)(1-u^2)^2 I_{[-1,1]}(u)$ and the Wiener error process where $R(s,t) = \min(s,t)$.

The weight density w, chosen here, is the uniform on $[0,1]$, i.e., $w \equiv 1$ on $[0,1]$, we consider the optimal bandwidth with respect to the exact IMSE of the two estimators, i.e., $\inf_{0<h<1} \text{IMSE}(h)$. The bandwidth h is chosen over a grid of 50 bandwidths from 0.09 to 0.5. The results are given in Tables 1, 2 and 3 for $n = 10, 30, 50$ and for different values of m. The tables present the integrated bias squared denoted by $Ibias^2$, integrated variance denoted by $Ivar$ and the IMSE together with the optimal bandwidth associated to the smallest exact IMSE for each estimator.

First, we can see from all these tables that, the optimal bandwidth decreases when m increases, as shown in Corollary 1. In addition, the optimal bandwidth of the projection estimator is smaller than that of the GM estimator for a large n.

It is shown also that both, the $Ivar$ and the $Ibias^2$, of the two estimators decrease when m increases. In addition, the GM estimator has a smaller $Ibias^2$ while the projection estimator has a smaller $Ivar$. The IMSE is slightly smaller for the projection estimator. These results are confirmed in Figure 3.

In Figure 4 when the regression function given by (M2) is considered, we have the same results mentioned above for the IMSE and $Ivar$, but with a smaller $Ibias^2$ for the projection estimator when m is large ($m \geq 10$). This can be argued by the fact that the bias depends very much on the second derivative of the regression function.

It should be noted here that, in order to solve the problem at the edges $[0,h] \cap [1-h,1]$, it was necessary to modify the kernel by using,

$$\tilde{K}(x) = K(x) \Big/ \int_{x-h}^{x+h} K(u)du, \quad \text{for} \quad x \in [0,h] \cap [1-h,1],$$

as suggested by [18].

Table 1: The integrated squared bias, integrated variance, IMSE and the optimal bandwidth for $n = 10$ and different values of m under the Wiener error process, for the GM and the projection estimators.

$n = 10$	m	$Ibias^2$	$Ivar$	IMSE	h_{opt}
GM	5	3.4410×10^{-3}	8.7513×10^{-2}	9.0954×10^{-2}	0.416
Pro		7.4427×10^{-3}	7.2007×10^{-2}	7.9450×10^{-2}	0.500
GM	15	8.10002×10^{-4}	3.0451×10^{-2}	3.1451×10^{-2}	0.299
Pro		23.401×10^{-4}	2.6937×10^{-2}	2.9277×10^{-2}	0.307
GM	30	4.2610×10^{-4}	1.5616×10^{-2}	1.6042×10^{-2}	0.232
Pro		8.7476×10^{-4}	1.4468×10^{-2}	1.5343×10^{-2}	0.198

Table 2: The integrated squared bias, integrated variance, IMSE and the optimal bandwidth for $n = 30$ and different values of m under the Wiener error process, for the GM and the projection estimators.

$n = 30$	m	$Ibias^2$	$Ivar$	IMSE	h_{opt}
GM	5	3.2965×10^{-3}	8.4542×10^{-2}	8.7838×10^{-2}	0.424
Pro		3.5847×10^{-3}	8.0784×10^{-2}	8.4368×10^{-2}	0.374
GM	15	8.6360×10^{-4}	2.9460×10^{-2}	3.0324×10^{-2}	0.307
Pro		16.907×10^{-4}	2.8055×10^{-2}	2.9746×10^{-2}	0.274
GM	30	3.6112×10^{-4}	1.5075×10^{-2}	1.5436×10^{-2}	0.248
Pro		8.0738×10^{-4}	1.4626×10^{-2}	1.5433×10^{-2}	0.198

Table 3: The integrated squared bias, integrated variance, IMSE and the optimal bandwidth for $n = 50$ and different values of m under the Wiener error process, for the GM and the projection estimators.

$n = 50$	m	$Ibias^2$	$Ivar$	IMSE	h_{opt}
GM	5	3.2822×10^{-3}	8.4325×10^{-2}	8.7607×10^{-2}	0.424
Pro		2.3635×10^{-3}	8.4423×10^{-2}	8.6787×10^{-2}	0.315
GM	15	8.5576×10^{-4}	2.9388×10^{-2}	3.0244×10^{-2}	0.307
Pro		12.413×10^{-4}	2.8845×10^{-2}	3.0086×10^{-2}	0.240
GM	30	3.5593×10^{-4}	1.5038×10^{-2}	1.5394×10^{-3}	0.248
Pro		6.6009×10^{-4}	1.4842×10^{-2}	1.5502×10^{-2}	0.182

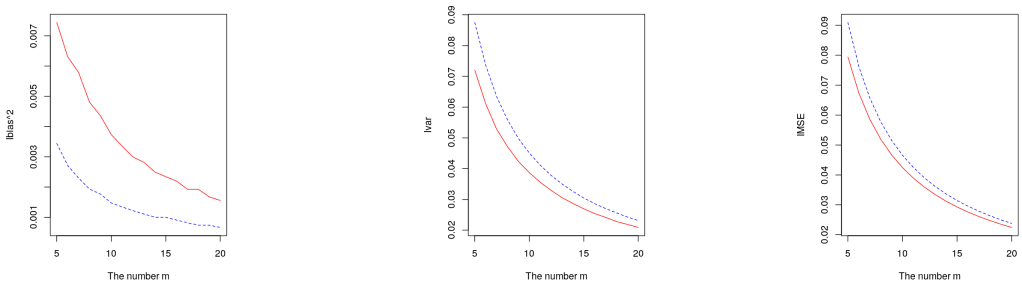

Figure 3: The $Ibias^2$, $Ivar$ and IMSE in term of m when $n = 10$ and $g(x)$ of model (M1) The projection estimator is in plain line and the GM estimator is in dashed line.

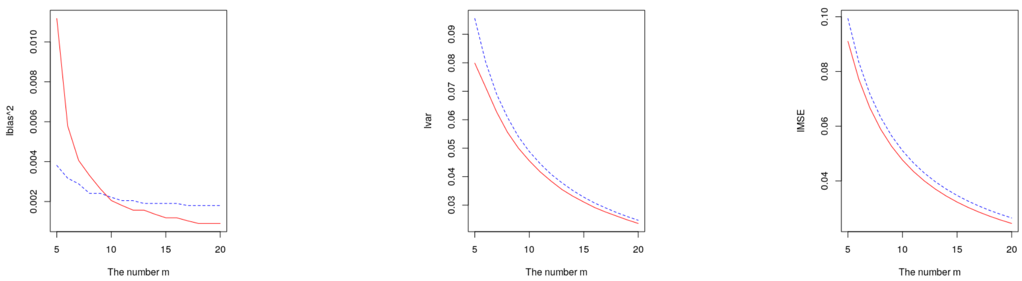

Figure 4: The $Ibias^2$, $Ivar$ and IMSE in term of m when $n = 10$ and $g(x)$ of model (M2). The projection estimator is in plain line and the GM estimator is in dashed line.

6 Proofs

In this section, we shall omit the index n in $t_{i,n}$ when there is no ambiguity.

6.1 Proof of Proposition 1.

It is known that (see, for instance [29] page 88) if $R(s,t) = \int_0^{\min(s,t)} u^\beta \, du$ then for any functions u and v and for any sampling design T_n we have,

$$u_{|T_n}{}' R_{|T_n}^{-1} v_{|T_n} = \frac{u(t_1)v(t_1)}{t_1^{\beta+1}} + \sum_{k=1}^{n-1} \frac{(u(t_{k+1}) - u(t_k))(v(t_{k+1}) - v(t_k))}{t_{k+1}^{\beta+1} - t_k^{\beta+1}}.$$

Replacing $u = f_{x,h}$ and $v = \overline{Y}$ we have,

$$\hat{g}_n^{pro}(x) = \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{t_1^{\beta+1}} + \sum_{i=1}^{n-1} \frac{(f_{x,h}(t_{i+1}) - f_{x,h}(t_i))(\overline{Y}(t_{i+1}) - \overline{Y}(t_i))}{t_{i+1}^{\beta+1} - t_i^{\beta+1}}. \tag{17}$$

Recall that $R(s,t) = \frac{1}{\beta+1} \min(s,t)^{\beta+1}$ and,

$$f_{x,h}(t_i) = \int_0^1 R(s,t_i)\varphi_{x,h}(s) \, ds = \frac{1}{\beta+1}\Big(\int_0^{t_i} s^{\beta+1} \, \varphi_{x,h}(s) \, ds + t_i^{\beta+1} \int_{t_i}^1 \varphi_{x,h}(s) \, ds \Big).$$

Thus,

$$f_{x,h}(t_{i+1}) - f_{x,h}(t_i) = \frac{1}{\beta+1}\Big(\int_0^{t_{i+1}} s^{\beta+1}\varphi_{x,h}(s)\,ds + t_{i+1}^{\beta+1}\int_{t_{i+1}}^1 \varphi_{x,h}(s)\,ds$$

$$-\int_0^{t_i} s^{\beta+1}\varphi_{x,h}(s)\,ds - t_i^{\beta+1}\int_{t_i}^1 \varphi_{x,h}(s)\,ds + t_{i+1}^{\beta+1}\int_{t_i}^1 \varphi_{x,h}(s)\,ds - t_{i+1}^{\beta+1}\int_{t_i}^1 \varphi_{x,h}(s)\,ds\Big)$$

$$= \frac{1}{\beta+1}\Big(\int_{t_i}^{t_{i+1}} (s^{\beta+1} - t_{i+1}^{\beta+1})\varphi_{x,h}(s)\,ds + (t_{i+1}^{\beta+1} - t_i^{\beta+1})\int_{t_i}^1 \varphi_{x,h}(s)\,ds\Big).$$

Thus,

$$\hat{g}_n^{pro}(x) = \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{t_1^{\beta+1}} + \frac{1}{\beta+1}\Big(\sum_{i=1}^{n-1}(\overline{Y}(t_{i+1}) - \overline{Y}(t_i))\int_{t_i}^1 \varphi_{x,h}(s)\,ds$$

$$+\sum_{i=1}^{n-1}\frac{\overline{Y}(t_{i+1}) - \overline{Y}(t_i)}{t_{i+1}^{\beta+1} - t_i^{\beta+1}}\int_{t_i}^{t_{i+1}} (s^{\beta+1} - t_{i+1}^{\beta+1})\varphi_{x,h}(s)\,ds\Big)$$

$$= \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{t_1^{\beta+1}} + \frac{1}{\beta+1}\Big(\sum_{i=2}^{n-1}\overline{Y}(t_i)\int_{t_{i-1}}^{t_i} \varphi_{x,h}(s)\,ds - \overline{Y}(t_1)\int_{t_1}^1 \varphi_{x,h}(s)\,ds$$

$$+\overline{Y}(t_n)\int_{t_{n-1}}^1 \varphi_{x,h}(s)\,ds + \sum_{i=1}^{n-1}\frac{\overline{Y}(t_{i+1}) - \overline{Y}(t_i)}{t_{i+1}^{\beta+1} - t_i^{\beta+1}}\int_{t_i}^{t_{i+1}} (s^{\beta+1} - t_{i+1}^{\beta+1})\varphi_{x,h}(s)\,ds\Big).$$

Letting $t_0 = \overline{Y}(t_0) = 0$ we have,

$$\frac{f_{x,h}(t_1)\overline{Y}(t_1)}{t_1^{\beta+1}} = \frac{1}{\beta+1}\Big(\frac{\overline{Y}(t_1)}{t_1^{\beta+1}}\int_0^{t_1} s^{\beta+1}\varphi_{x,h}(s)\,ds + \overline{Y}(t_1)\int_{t_1}^1 \varphi_{x,h}(s)\,ds\Big)$$

$$= \frac{1}{\beta+1}\Big(\frac{\overline{Y}(t_1) - \overline{Y}(t_0)}{t_1^{\beta+1} - t_0^{\beta+1}}\int_0^{t_1} (s^{\beta+1} - t_1^{\beta+1})\varphi_{x,h}(s)\,ds$$

$$+\overline{Y}(t_1)\int_0^{t_1} \varphi_{x,h}(s)\,ds + \overline{Y}(t_1)\int_{t_1}^1 \varphi_{x,h}(s)\,ds\Big).$$

Finally,

$$\hat{g}_n^{pro}(x) = \frac{1}{\beta+1}\Big(\sum_{i=2}^{n-1}\overline{Y}(t_i)\int_{t_{i-1}}^{t_i} \varphi_{x,h}(s)\,ds - \overline{Y}(t_1)\int_{t_1}^1 \varphi_{x,h}(s)\,ds + \overline{Y}(t_n)\int_{t_{n-1}}^1 \varphi_{x,h}(s)\,ds$$

$$+\overline{Y}(t_n)\int_{t_{n-1}}^{t_n} \varphi_{x,h}(s)\,ds - \overline{Y}(t_n)\int_{t_{n-1}}^{t_n} \varphi_{x,h}(s)\,ds$$

$$+\sum_{i=1}^{n-1}\frac{\overline{Y}(t_{i+1}) - \overline{Y}(t_i)}{t_{i+1}^{\beta+1} - t_i^{\beta+1}}\int_{t_i}^{t_{i+1}} (s^{\beta+1} - t_{i+1}^{\beta+1})\varphi_{x,h}(s)\,ds$$

$$+\frac{\overline{Y}(t_1) - \overline{Y}(t_0)}{t_1^{\beta+1} - t_0^{\beta+1}}\int_0^{t_1} (s^{\beta+1} - t_1^{\beta+1})\varphi_{x,h}(s)\,ds + \overline{Y}(t_1)\int_0^{t_1} \varphi_{x,h}(s)\,ds + \overline{Y}(t_1)\int_{t_1}^1 \varphi_{x,h}(s)\,ds\Big)$$

$$= \frac{1}{\beta+1}\Big(\sum_{i=1}^{n+1}\overline{Y}(t_i)\int_{t_{i-1}}^{t_i} \varphi_{x,h}(s)\,ds + \sum_{i=0}^{n-1}\frac{\overline{Y}(t_{i+1}) - \overline{Y}(t_i)}{t_{i+1}^{\beta+1} - t_i^{\beta+1}}\int_{t_i}^{t_{i+1}} (s^{\beta+1} - t_{i+1}^{\beta+1})\varphi_{x,h}(s)\,ds\Big),$$

where $t_{n+1} = 1$ and $\overline{Y}(t_{n+1}) = \overline{Y}(t_n)$. This concludes the proof of Proposition 1. $\square$

6.2 Proof of Proposition 2.

It is known (see [2] page 210) that for every functions u and v and for every design T_n we have,

$$
\begin{aligned}
u'_{|T_n} R^{-1}_{|T_n} v_{|T_n} = {} & \frac{u(t_1)v(t_1)}{1-e^{-2(t_2-t_1)}} + \frac{u(t_n)v(t_n)}{1-e^{-2(t_n-t_{n-1})}} + \sum_{i=2}^{n-1} \frac{u(t_i)v(t_i)(1-e^{-2(t_{i+1}-t_{i-1})})}{(1-e^{-2(t_{i+1}-t_i)})(1-e^{-2(t_i-t_{i-1})})} \\
& - \sum_{i=1}^{n-1} \frac{u(t_i)v(t_{i+1}) + u(t_{i+1})v(t_i)}{1-e^{-2(t_{i+1}-t_i)}} e^{-(t_{i+1}-t_i)}.
\end{aligned}
$$

Taking $u = f_{x,h}$ and $v = \overline{Y}$ we get,

$$
\begin{aligned}
\hat{g}_n^{pro}(x) = {} & \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}} + \frac{f_{x,h}(t_n)\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}} + \sum_{i=2}^{n-1} \frac{f_{x,h}(t_i)\overline{Y}(t_i)(1-e^{-2(t_{i+1}-t_{i-1})})}{(1-e^{-2(t_{i+1}-t_i)})(1-e^{-2(t_i-t_{i-1})})} \\
& - \sum_{i=1}^{n-1} \frac{f_{x,h}(t_i)\overline{Y}(t_{i+1}) + f_{x,h}(t_{i+1})\overline{Y}(t_i)}{1-e^{-2(t_{i+1}-t_i)}} e^{-(t_{i+1}-t_i)} \\
\triangleq {} & \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}} + \frac{f_{x,h}(t_n)\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}} + A. \qquad (18)
\end{aligned}
$$

Note that,

$$
1-e^{-2(t_{i+1}-t_{i-1})} = (1-e^{-2(t_{i+1}-t_i)}) + (1-e^{-2(t_i-t_{i-1})}) - (1-e^{-2(t_i-t_{i-1})})(1-e^{-2(t_{i+1}-t_i)}).
$$

Thus,

$$
\begin{aligned}
A = {} & \sum_{i=2}^{n-1} \frac{f_{x,h}(t_i)\overline{Y}(t_i)}{1-e^{-2(t_i-t_{i-1})}} + \sum_{i=2}^{n-1} \frac{f_{x,h}(t_i)\overline{Y}(t_i)}{1-e^{-2(t_{i+1}-t_i)}} - \sum_{i=2}^{n-1} f_{x,h}(t_i)\overline{Y}(t_i) \\
& - \sum_{i=2}^{n} \frac{f_{x,h}(t_{i-1})\overline{Y}(t_i)}{1-e^{-2(t_i-t_{i-1})}} e^{-(t_i-t_{i-1})} - \sum_{i=1}^{n-1} \frac{f_{x,h}(t_{i+1})\overline{Y}(t_i)}{1-e^{-2(t_{i+1}-t_i)}} e^{-(t_{i+1}-t_i)} \\
= {} & \sum_{i=2}^{n-1} \frac{\overline{Y}(t_i)}{1-e^{-2(t_i-t_{i-1})}} \Big(f_{x,h}(t_i) - f_{x,h}(t_{i-1}) e^{-(t_i-t_{i-1})} \Big) - \frac{f_{x,h}(t_{n-1})\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}} e^{-(t_n-t_{n-1})} \\
& + \sum_{i=2}^{n-1} \frac{\overline{Y}(t_i)}{1-e^{-2(t_{i+1}-t_i)}} \Big(f_{x,h}(t_i) - f_{x,h}(t_{i+1}) e^{-(t_{i+1}-t_i)} \Big) - \frac{f_{x,h}(t_2)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}} e^{-(t_2-t_1)} \\
& - \sum_{i=2}^{n-1} f_{x,h}(t_i)\overline{Y}(t_i) \qquad (19)
\end{aligned}
$$

Simple calculations yield,

$$
\begin{aligned}
& f_{x,h}(t_i) - f_{x,h}(t_{i-1}) e^{-(t_i-t_{i-1})} = \\
& e^{-t_i} \int_{t_{i-1}}^{t_i} e^{s} \varphi_{x,h}(s)\, ds - e^{t_i} \int_{t_{i-1}}^{t_i} e^{-s} \varphi_{x,h}(s)\, ds + e^{t_i}(1-e^{-2(t_i-t_{i-1})}) \int_{t_{i-1}}^{1} e^{-s} \varphi_{x,h}(s)\, ds. \qquad (20)
\end{aligned}
$$

In the same way we have,

$$f_{x,h}(t_i) - f_{x,h}(t_{i+1})e^{-(t_{i+1}-t_i)} =$$
$$e^{t_i}\int_{t_i}^{t_{i+1}} e^{-s}\varphi_{x,h}(s)\,ds - e^{-t_i}\int_{t_i}^{t_{i+1}} e^{s}\varphi_{x,h}(s)\,ds + e^{-t_i}(1-e^{-2(t_{i+1}-t_i)})\int_{0}^{t_{i+1}} e^{s}\varphi_{x,h}(s)\,ds. \tag{21}$$

It is easy to verify that,

$$\sum_{i=2}^{n-1} f_{x,h}(t_i)\overline{Y}(t_i) = \sum_{i=2}^{n-1}\overline{Y}(t_i)e^{-t_i}\int_0^{t_i} e^{s}\varphi_{x,h}(s)\,ds + \sum_{i=2}^{n-1}\overline{Y}(t_i)e^{t_i}\int_{t_i}^{1} e^{-s}\varphi_{x,h}(s)\,ds. \tag{22}$$

We obtain using Equations (19), (20), (21) and (22),

$$\begin{aligned}
A =& \sum_{i=2}^{n-1}\overline{Y}(t_i)e^{t_i}\int_{t_{i-1}}^{1} e^{-s}\varphi_{x,h}(s)\,ds + \sum_{i=2}^{n-1}\frac{\overline{Y}(t_i)e^{-t_i}}{1-e^{-2(t_i-t_{i-1})}}\int_{t_{i-1}}^{t_i} e^{s}\varphi_{x,h}(s)\,ds \\
&- \sum_{i=2}^{n-1}\frac{\overline{Y}(t_i)e^{t_i}}{1-e^{-2(t_i-t_{i-1})}}\int_{t_{i-1}}^{t_i} e^{-s}\varphi_{x,h}(s)\,ds + \sum_{i=2}^{n-1}\overline{Y}(t_i)e^{-t_i}\int_0^{t_{i+1}} e^{s}\varphi_{x,h}(s)\,ds \\
&+ \sum_{i=2}^{n-1}\frac{\overline{Y}(t_i)e^{t_i}}{1-e^{-2(t_{i+1}-t_i)}}\int_{t_i}^{t_{i+1}} e^{-s}\varphi_{x,h}(s)\,ds - \sum_{i=2}^{n-1}\frac{\overline{Y}(t_i)e^{-t_i}}{1-e^{-2(t_{i+1}-t_i)}}\int_{t_i}^{t_{i+1}} e^{s}\varphi_{x,h}(s)\,ds \\
&- \frac{f_{x,h}(t_2)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}}e^{-(t_2-t_1)} - \frac{f_{x,h}(t_{n-1})\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}}e^{-(t_n-t_{n-1})} \\
&- \sum_{i=2}^{n-1}\overline{Y}(t_i)e^{-t_i}\int_0^{t_i} e^{s}\varphi_{x,h}(s)\,ds - \sum_{i=2}^{n-1}\overline{Y}(t_i)e^{t_i}\int_{t_i}^{1} e^{-s}\varphi_{x,h}(s)\,ds.
\end{aligned}$$

Replacing this expression of A in (18) gives,

$$\begin{aligned}
\hat{g}_n^{pro}(x) =& \sum_{i=2}^{n-1}\overline{Y}(t_i)\int_{t_{i-1}}^{t_{i+1}} e^{|t_i-s|}\varphi_{x,h}(s)\,ds + \sum_{i=2}^{n-2}\frac{\overline{Y}(t_{i+1})e^{-t_{i+1}} - \overline{Y}(t_i)e^{-t_i}}{1-e^{-2(t_{i+1}-t_i)}}\int_{t_i}^{t_{i+1}} e^{s}\varphi_{x,h}(s)\,ds \\
&- \sum_{i=2}^{n-2}\frac{\overline{Y}(t_{i+1})e^{t_{i+1}} - \overline{Y}(t_i)e^{t_i}}{1-e^{-2(t_{i+1}-t_i)}}\int_{t_i}^{t_{i+1}} e^{-s}\varphi_{x,h}(s)\,ds + \frac{\overline{Y}(t_2)e^{-t_2}}{1-e^{-2(t_2-t_1)}}\int_{t_1}^{t_2} e^{s}\varphi_{x,h}(s)\,ds \\
&- \frac{\overline{Y}(t_{n-1})e^{-t_{n-1}}}{1-e^{-2(t_n-t_{n-1})}}\int_{t_{n-1}}^{t_n} e^{s}\varphi_{x,h}(s)\,ds - \frac{\overline{Y}(t_2)e^{t_2}}{1-e^{-2(t_2-t_1)}}\int_{t_1}^{t_2} e^{-s}\varphi_{x,h}(s)\,ds \\
&+ \frac{\overline{Y}(t_{n-1})e^{t_{n-1}}}{1-e^{-2(t_n-t_{n-1})}}\int_{t_{n-1}}^{t_n} e^{-s}\varphi_{x,h}(s)\,ds + \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}} - \frac{f_{x,h}(t_2)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}}e^{-(t_2-t_1)} \\
&+ \frac{f_{x,h}(t_n)\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}} - \frac{f_{x,h}(t_{n-1})\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}}e^{-(t_n-t_{n-1})}.
\end{aligned} \tag{23}$$

Note that Equation (21) yields,

$$\begin{aligned}
\frac{\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}}\left(f_{x,h}(t_1) - f_{x,h}(t_2)e^{-(t_2-t_1)}\right) =& \frac{\overline{Y}(t_1)e^{t_1}}{1-e^{-2(t_2-t_1)}}\int_{t_1}^{t_2} e^{-s}\varphi_{x,h}(s)\,ds \\
&- \frac{\overline{Y}(t_1)e^{-t_1}}{1-e^{-2(t_2-t_1)}}\int_{t_1}^{t_2} e^{s}\varphi_{x,h}(s)\,ds + \overline{Y}(t_1)e^{-t_1}\int_{t_1}^{t_2} e^{s}\varphi_{x,h}(s)\,ds.
\end{aligned} \tag{24}$$

Similarly, Equation (20) yields,

$$\frac{\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}}\big(f_{x,h}(t_n)-f_{x,h}(t_{n-1})e^{-(t_n-t_{n-1})}\big)=\frac{\overline{Y}(t_n)e^{-t_n}}{1-e^{-2(t_n-t_{n-1})}}\int_{t_{n-1}}^{t_n}e^{s}\varphi_{x,h}(s)\,ds$$
$$-\frac{\overline{Y}(t_n)e^{t_n}}{1-e^{-2(t_n-t_{n-1})}}\int_{t_{n-1}}^{t_n}e^{-s}\varphi_{x,h}(s)\,ds+\overline{Y}(t_n)e^{t_n}\int_{t_{n-1}}^{1}e^{-s}\varphi_{x,h}(s)\,ds. \tag{25}$$

We obtain using (24) and (25) in (23),

$$\begin{aligned}
\hat{g}_n^{pro}(x)&=\sum_{i=2}^{n-1}\overline{Y}(t_i)\int_{t_{i-1}}^{t_{i+1}}e^{|t_i-s|}\varphi_{x,h}(s)\,ds+\sum_{i=2}^{n-2}\frac{\overline{Y}(t_{i+1})e^{-t_{i+1}}-\overline{Y}(t_i)e^{-t_i}}{1-e^{-2(t_{i+1}-t_i)}}\int_{t_i}^{t_{i+1}}e^{s}\varphi_{x,h}(s)\,ds\\
&-\sum_{i=2}^{n-2}\frac{\overline{Y}(t_{i+1})e^{t_{i+1}}-\overline{Y}(t_i)e^{t_i}}{1-e^{-2(t_{i+1}-t_i)}}\int_{t_i}^{t_{i+1}}e^{-s}\varphi_{x,h}(s)\,ds+\frac{\overline{Y}(t_2)e^{-t_2}}{1-e^{-2(t_2-t_1)}}\int_{t_1}^{t_2}e^{s}\varphi_{x,h}(s)\,ds\\
&-\frac{\overline{Y}(t_{n-1})e^{-t_{n-1}}}{1-e^{-2(t_n-t_{n-1})}}\int_{t_{n-1}}^{t_n}e^{s}\varphi_{x,h}(s)\,ds-\frac{\overline{Y}(t_2)e^{t_2}}{1-e^{-2(t_2-t_1)}}\int_{t_1}^{t_2}e^{-s}\varphi_{x,h}(s)\,ds\\
&+\frac{\overline{Y}(t_{n-1})e^{t_{n-1}}}{1-e^{-2(t_n-t_{n-1})}}\int_{t_{n-1}}^{t_n}e^{-s}\varphi_{x,h}(s)\,ds+\frac{\overline{Y}(t_1)e^{t_1}}{1-e^{-2(t_2-t_1)}}\int_{t_1}^{t_2}e^{-s}\varphi_{x,h}(s)\,ds\\
&-\frac{\overline{Y}(t_1)e^{-t_1}}{1-e^{-2(t_2-t_1)}}\int_{t_1}^{t_2}e^{s}\varphi_{x,h}(s)\,ds+\overline{Y}(t_1)e^{-t_1}\int_{t_1}^{t_2}e^{s}\varphi_{x,h}(s)\,ds\\
&+\frac{\overline{Y}(t_n)e^{-t_n}}{1-e^{-2(t_n-t_{n-1})}}\int_{t_{n-1}}^{t_n}e^{s}\varphi_{x,h}(s)\,ds-\frac{\overline{Y}(t_n)e^{t_n}}{1-e^{-2(t_n-t_{n-1})}}\int_{t_{n-1}}^{t_n}e^{-s}\varphi_{x,h}(s)\,ds\\
&+\overline{Y}(t_n)e^{t_n}\int_{t_{n-1}}^{1}e^{-s}\varphi_{x,h}(s)\,ds
\end{aligned}$$

$$\begin{aligned}
&=\sum_{i=2}^{n-1}\overline{Y}(t_i)\int_{t_{i-1}}^{t_{i+1}}e^{|s-t_i|}\varphi_{x,h}(s)\,ds+\overline{Y}(t_1)\int_0^{t_2}e^{s-t_1}\varphi_{x,h}(s)\,ds+\overline{Y}(t_n)\int_{t_{n-1}}^{1}e^{t_n-s}\varphi_{x,h}(s)\,ds\\
&-\sum_{i=1}^{n-1}\frac{e^{t_{i+1}}\overline{Y}(t_{i+1})-e^{t_i}\overline{Y}(t_i)}{1-e^{-2(t_{i+1}-t_i)}}\int_{t_i}^{t_{i+1}}e^{-s}\varphi_{x,h}(s)\,ds\\
&+\sum_{i=1}^{n-1}\frac{e^{-t_{i+1}}\overline{Y}(t_{i+1})-e^{-t_i}\overline{Y}(t_i)}{1-e^{-2(t_{i+1}-t_i)}}\int_{t_i}^{t_{i+1}}e^{s}\varphi_{x,h}(s)\,ds.
\end{aligned}$$

This concludes the proof of Proposition 2. □

6.3 Proof of Lemma 1.

Let $(u,v)\in[-1,1]^2$. We first consider the triangle $\{-1<u<v<1\}$ which is further split into smaller triangles:

$$D_1=\{0<u<v<1\},\ D_2=\{-1<u<0<v<1\}\ \text{ and }\ D_3=\{-1<u<v<0\}.$$

Let $b\in]0,1[$. For $(u,v)\in D_1$, using Assumption (A), Taylor expansion of R around (x,x) gives,

$$\begin{aligned}
R(x+bu,x+bv)&=R(x,x+bv)+buR^{(1,0)}(x,x+bv)+\frac{1}{2}b^2u^2R^{(2,0)}(\varepsilon_x,x+bv)\\
&=R(x,x)+bvR^{(0,1)}(x,\eta_x)+buR^{(1,0)}(x,x+bv)+\frac{1}{2}b^2u^2R^{(2,0)}(\varepsilon_x,x+bv),
\end{aligned}$$

where $x < \varepsilon_x < x + bu < x + bv$ and $x < \eta_x < x + bv$. Thus,

$$R(x+bu, x+bv) = R(x,x) + bvR^{(0,1)}(x, x^+) + buR^{(0,1)}(x, x^-) + o(b).$$

Now, for $(u,v) \in D_2$ we obtain in the same way,

$$\begin{aligned} R(x+bu, x+bv) &= R(x, x+bv) + buR^{(1,0)}(x, x+bv) + \frac{1}{2}b^2u^2R^{(2,0)}(\varepsilon_x, x+bv) \\ &= R(x,x) + bvR^{(0,1)}(x, \eta_x) + buR^{(1,0)}(x, x+bv) + \frac{1}{2}b^2u^2R^{(2,0)}(\varepsilon_x, x+bv), \end{aligned}$$

where $x + bu < \varepsilon_x < x < x + bv$ and $x < \eta_x < x + bv$. Thus,

$$R(x+bu, x+bv) = R(x,x) + bvR^{(0,1)}(x, x^+) + buR^{(0,1)}(x, x^-) + o(b).$$

Finally, for $(u,v) \in D_3$ we get,

$$\begin{aligned} R(x+bu, x+bv) &= R(x+bu, x) + bvR^{(0,1)}(x+bu, x) + \frac{1}{2}b^2v^2R^{(0,2)}(x+bu, \eta_x) \\ &= R(x,x) + ubR^{(1,0)}(\varepsilon_x, x) + bvR^{(0,1)}(x+bu, \eta_x) + \frac{1}{2}b^2v^2R^{(0,2)}(x+bu, \eta_x), \end{aligned}$$

where $x + hu < x + bv < \eta_x < x$ and $x + bu < \varepsilon_x < x$. Thus,

$$R(x+bu, x+bv) = R(x,x) + bvR^{(0,1)}(x, x^+) + buR^{(0,1)}(x, x^-) + o(b).$$

Hence for $v > u$ we have,

$$\begin{aligned} R(x+bu, x+bv) = R(x,x) &+ \frac{b}{2}\big(R^{(0,1)}(x, x^+) + R^{(0,1)}(x, x^-)\big)(u+v) \\ &+ \frac{b}{2}\big(R^{(0,1)}(x, x^+) - R^{(0,1)}(x, x^-)\big)(v-u) + o(b). \end{aligned}$$

Similarly, we obtain for the triangular $\{1 > u > v > -1\}$,

$$\begin{aligned} R(x+bu, x+bv) = R(x,x) &+ \frac{b}{2}\big(R^{(0,1)}(x, x^+) + R^{(0,1)}(x, x^-)\big)(u+v) \\ &+ \frac{b}{2}\big(R^{(0,1)}(x, x^+) - R^{(0,1)}(x, x^-)\big)(u-v). \end{aligned}$$

Thus, for $(u,v) \in [-1,1]^2$ we have,

$$\begin{aligned} R(x+bu, x+bv) = R(x,x) &+ \frac{b}{2}\big(R^{(0,1)}(x, x^+) + R^{(0,1)}(x, x^-)\big)(u+v) \\ &+ \frac{b}{2}\big(R^{(0,1)}(x, x^+) - R^{(0,1)}(x, x^-)\big)|u-v|. \end{aligned} \tag{26}$$

Consider now a function g, bounded and integrable on $[-1,1]$. The Dominated Convergence Theorem yields that $R(., t) \times g$ is an integrable function for every $t \in [-1,1]$. Using (26) and putting,

$$\gamma(x) = \frac{1}{2}\big(R^{(0,1)}(x, x^+) + R^{(0,1)}(x, x^-)\big),$$

we obtain,

$$\begin{aligned}\iint_{[-1,1]^2} R(x+bu,x+bv)g(u)g(v)dudv &= R(x,x)\left(\int_{-1}^{1} g(u)du\right)^2 \\ &+ 2\gamma(x)b\int_{-1}^{1} g(u)du \int_{-1}^{1} vg(v)dv - \frac{b}{2}\alpha(x)\iint_{[-1,1]^2} g(u)g(v)|u-v|dudv + o(b).\end{aligned} \tag{27}$$

The left side of (27) is non-negative since the autocovariance function R is a non-negative definite function. Taking $g(u) = u1_{[-1,1]}(u)$ we obtain,

$$\int_{-1}^{1} g(u)du = 0 \quad \text{and} \quad \iint_{[-1,1]^2} uv|u-v|dudv = -\frac{8}{15}.$$

Thus,

$$\frac{4}{15}\alpha(x) + o(b) \geq 0.$$

Taking b small enough concludes the proof of Lemma 1. □

6.4 Proof of Lemma 3.

The great lines of this proof are based on the work of [25] (c.f. Lemma 3.2 there). Let $x, h \in]0,1[$ and put $g_n = P_{T_n} f_{x,h}$, it is shown by (106) in the Appendix that,

$$g_n(t_i) = \sum_{j=1}^{n} m_{x,h}(t_j)R(t_j,t_i) \quad \text{for all } i = 1,\cdots,n.$$

On the one hand, Assumption (A) yields that g_n is twice differentiable on $[0,1]$ except on T_n, but it has left and right derivatives. Thus, for every $i = 1,\ldots,n$ we have,

$$g_n'(t_i^-) = \sum_{j=1}^{n} m_{x,h}(t_j)R^{(0,1)}(t_j,t_i^-) \quad \text{and} \quad g_n'(t_i^+) = \sum_{j=1}^{n} m_{x,h}(t_j)R^{(0,1)}(t_j,t_i^+).$$

Since for $j \neq i$, $R^{(0,1)}(t_j,t_i^-) = R^{(0,1)}(t_j,t_i^+)$ then Assumption (B) yields,

$$g_n'(t_i^-) - g_n'(t_i^+) = \alpha(t_i)m_{x,h}(t_i). \tag{28}$$

On the other hand, Assumption (A) yields that $f_{x,h}$ (as defined by (2)) is twice differentiable on $]0,1[$, thus for $i = 1,\ldots,n-1$, Taylor expansion of $f_{x,h} - g_n$ around t_i gives,

$$f_{x,h}(t_{i+1}) - g_n(t_{i+1}) = (f_{x,h}(t_i) - g_n(t_i)) + d_i(f_{x,h}'(t_i) - g_n'(t_i^+)) + \frac{1}{2}d_i^2(f_{x,h}''(\sigma_i) - g_n''(\sigma_i)),$$

where $d_i = t_{i+1} - t_i$ and $\sigma_i \in]t_i, t_{i+1}[$. Recall that, for all $i = 1,\ldots,n$, $f_{x,h}(t_i) = g_n(t_i)$ (see Appendix, Equation (104)). Thus,

$$f_{x,h}'(t_i) - g_n'(t_i^+) = -\frac{1}{2}d_i(f_{x,h}''(\sigma_i) - g_n''(\sigma_i)), \tag{29}$$

Similarly, for $i = 2,\ldots,n$, we have,

$$f_{x,h}'(t_i) - g_n'(t_i^-) = \frac{1}{2}d_{i-1}(f_{x,h}''(\theta_i) - g_n''(\theta_i)), \tag{30}$$

for some $\theta_i \in]t_{i-1}, t_i[$. We obtain subtracting (30) from (29) and using (28) for $i = 2, \ldots, n-1$,

$$\alpha(t_i)m_{x,h}(t_i) = -\frac{1}{2}d_i(f''_{x,h}(\sigma_i) - g''_n(\sigma_i)) - \frac{1}{2}d_{i-1}(f''_{x,h}(\theta_i) - g''_n(\theta_i)). \tag{31}$$

We shall now control the last expression. On the one hand we have,

$$f'_{x,h}(t) = \int_0^t R^{(0,1)}(s,t^+)\varphi_{x,h}(s)\, ds + \int_t^1 R^{(0,1)}(s,t^-)\varphi_{x,h}(s)\, ds, \tag{32}$$

and,

$$\begin{aligned} f''_{x,h}(t) = (R^{(0,1)}(t,t^+) - R^{(0,1)}(t,t^-))\varphi_{x,h}(t) + \int_0^1 R^{(0,2)}(s,t^+)\varphi_{x,h}(s)\, ds \\ - \alpha(t)\varphi_{x,h}(t) + \int_0^1 R^{(0,2)}(s,t^+)\varphi_{x,h}(s)\, ds. \end{aligned} \tag{33}$$

On the other hand we know, using (F3) in the Appendix, that every function in the RKHS(R), noted by $\mathcal{F}(\varepsilon)$, is continuous, hence Assumption (C) implies that $R^{(0,2)}(\cdot, t^+)$ is a continuous function on $[0,1]$ for every fixed $t \in [0,1]$. Thus,

$$R^{(0,2)}(t,t^+) = \lim_{s\downarrow t} R^{(0,2)}(s,t^+) = \lim_{s\downarrow t} R^{(0,2)}(s,t^-) = R^{(0,2)}(t,t^-),$$

from which we get that $R^{(0,2)}(t,t)$ exists. Hence for $i = 1, \ldots, n$ we have,

$$g''_n(t_i^-) = g''_n(t_i^+) = \sum_{j=1}^n m_{x,h}(t_j)R^{(0,2)}(t_j,t_i). \tag{34}$$

In addition, it is shown by (F4) in the Appendix that for every $t \in [0,1]$,

$$f''_{x,h}(t) - g''_n(t) = -\alpha(t)\varphi_{x,h}(t) + \langle R^{(0,2)}(\cdot,t), f_{x,h} - g_n\rangle, \tag{35}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{F}(\varepsilon)$. Injecting (35) in (31) we obtain,

$$\begin{aligned} \alpha(t_i)m_{x,h}(t_i) = \frac{1}{2}d_i\alpha(\sigma_i)\varphi_{x,h}(\sigma_i) + \frac{1}{2}d_{i-1}\alpha(\theta_i)\varphi_{x,h}(\theta_i) - \frac{1}{2}d_i\langle R^{(0,2)}(\cdot,\sigma_i), f_{x,h} - g_n\rangle \\ - \frac{1}{2}d_{i-1}\langle R^{(0,2)}(\cdot,\theta_i), f_{x,h} - g_n\rangle. \end{aligned}$$

Using Assumption (B) we obtain for $i = 2, \ldots, n-1$,

$$\begin{aligned} m_{x,h}(t_i) &= \frac{1}{2}(d_i + d_{i-1})\varphi_{x,h}(t_i) \ + \frac{1}{2\alpha(t_i)}d_i\big(\alpha(\sigma_i)\varphi_{x,h}(\sigma_i) - \alpha(t_i)\varphi_{x,h}(t_i)\big) \\ &+ \frac{1}{2\alpha(t_i)}d_{i-1}\big(\alpha(\theta_i)\varphi_{x,h}(\theta_i) - \alpha(t_i)\varphi_{x,h}(t_i)\big) - \frac{1}{2\alpha(t_i)}d_i\langle R^{(0,2)}(\cdot,\sigma_i), f_{x,h} - g_n\rangle \\ &- \frac{1}{2\alpha(t_i)}d_{i-1}\langle R^{(0,2)}(\cdot,\theta_i), f_{x,h} - g_n\rangle \\ &\triangleq \frac{1}{2}(d_i + d_{i-1})\varphi_{x,h}(t_i) \ + A_i^{(1)} + A_i^{(2)} - A_i^{(3)} - A_i^{(4)}, \end{aligned} \tag{36}$$

Using the Cauchy-Schwartz inequality, Assumption (C) and Equation (51) (in the proof of Proposition 5 below) we obtain,

$$|A_i^{(3)} + A_i^{(4)}| \leq \sup_{0\leq t\leq 1} \frac{1}{2\alpha(t)} ||R^{(0,2)}(.,t)|| \frac{\sqrt{C}}{\sqrt{h}} \sup_{0\leq j\leq n} d_j^2 \triangleq \beta_{n,h}, \tag{37}$$

where C is a positive constant defined in Proposition 5 below.
Recall that $\varphi_{x,h}$ is of support $[x-h, x+h]$, thus for t_i such that $[t_{i-1}, t_{i+1}] \cap]x-h, x+h[= \emptyset$, $\varphi_{x,h}(t) = 0$ so that $A_i^{(1)} = 0$ and $A_i^{(2)} = 0$. For t_i such that $[t_{i-1}, t_{i+1}] \cap]x-h, x+h[\neq \emptyset$, let,

$$\alpha_{n,h} = \sup_{0\leq i\leq n} \sup_{t_i\leq s,t\leq t_{i+1}} \frac{1}{2\alpha(t)} d_i |\alpha(s)\varphi_{x,h}(s) - \alpha(t)\varphi_{x,h}(t)|. \tag{38}$$

We obtain using (37) and (38) together with (36) for $i = 2, \ldots, n-1$,

$$m_{x,h}(t_i) = \begin{cases} \frac{1}{2}\varphi_{x,h}(t_i)(t_{i+1} - t_{i-1}) \ + \ O\big(\alpha_{n,h} + \beta_{n,h}\big) & if \ [t_{i-1}, t_{i+1}] \cap]x-h, x+h[\neq \emptyset \\ O\big(\beta_{n,h}\big) & otherwise. \end{cases}$$

After having obtained $m_{x,h}(t_i)$ for $i = 2, \ldots, n-1$, we are now able to obtain $m_{x,h}(t_1)$ and $m_{x,h}(t_n)$. We have for $i = 1, \ldots, n$,

$$R(t_1, t_i)m_{x,h}(t_1) + R(t_n, t_i)m_{x,h}(t_n) = f_{x,h}(t_i) - \sum_{j=2}^{n-1} m_{x,h}(t_j)R(t_j, t_i). \tag{39}$$

Recall that $N_{T_n} = \text{Card } I_{x,h} = \text{Card } \{i = 1, \cdots, n \ : \ [t_{i-1}, t_{i+1}] \cap]x-h, x+h[\neq \emptyset\}$ and that $t_{x,i}$ are the points of T_n for which $i \in I_{x,h}$. We have,

$$\sum_{j=2}^{n-1} m_{x,h}(t_j)R(t_j, t_i) = \sum_{j=1}^{N_{T_n}} m_{x,h}(t_{x,j})R(t_{x,j}, t_i) + \sum_{j=2}^{n-1} 1_{\{j\notin I_{x,h}\}} m_{x,h}(t_j)R(t_j, t_i).$$

On the one hand, we have using (36) (where $A_{x,j}$ stands for A_j with t_j replaced by $t_{x,j}$),

$$\begin{aligned} \sum_{j=2}^{n-1} m_{x,h}(t_j)R(t_j, t_i) &= \frac{1}{2}\sum_{j=1}^{N_{T_n}} (d_{x,j} + d_{x,j-1})\varphi_{x,h}(t_{x,j})R(t_{x,j}, t_i) \\ &+ \sum_{j=1}^{N_{T_n}} (A_{x,j}^1 + A_{x,j}^2 - A_{x,j}^3 - A_{x,j}^4)R(t_{x,j}, t_i) - \sum_{j=2}^{n-1} 1_{\{j\notin I_{x,h}\}}(A_j^3 + A_j^4)R(t_j, t_i) \\ &= \frac{1}{2}\sum_{j=1}^{N_{T_n}} (d_{x,j} + d_{x,j-1})\varphi_{x,h}(t_{x,j})R(t_{x,j}, t_i) + \sum_{j=1}^{N_{T_n}} (A_{x,j}^1 + A_{x,j}^2)R(t_{x,j}, t_i) - \sum_{j=1}^{n} (A_j^3 + A_j^4)R(t_j, t_i). \end{aligned} \tag{40}$$

On the other hand,

$$\begin{aligned} f_{x,h}(t_i) &= \int_0^1 R(s, t_i)\varphi_{x,h}(s)\, ds = \int_{x-h}^{x+h} R(s, t_i)\varphi_{x,h}(s)\, ds = \frac{1}{2}\sum_{j=1}^{N_{T_n}} \int_{t_{x,j-1}}^{t_{x,j+1}} R(s, t_i)\varphi_{x,h}(s)\, ds \\ &= \frac{1}{2}\sum_{j=1}^{N_{T_n}} (d_{x,j} + d_{x,j-1})R(t_{x,j}, t_i)\varphi_{x,h}(t_j) + \frac{1}{2}\sum_{j=1}^{N_{T_n}} \int_{t_{x,j-1}}^{t_{x,j+1}} (R(s, t_i)\varphi_{x,h}(s) - R(t_{x,j}, t_i)\varphi_{x,h}(t_{x,j}))\, ds. \end{aligned} \tag{41}$$

Inserting (40) and (41) in (39) we obtain for $i = 1, \ldots, n$,

$$R(t_1, t_i)m_{x,h}(t_1) + R(t_n, t_i)m_{x,h}(t_n) = \frac{1}{2}\sum_{j=1}^{N_{T_n}} \int_{t_{x,j-1}}^{t_{x,j+1}} (R(s, t_i)\varphi_{x,h}(s) - R(t_{x,j}, t_i)\varphi_{x,h}(t_{x,j}))\, ds$$

$$-\sum_{j=1}^{N_{T_n}}(A^1_{x,j} + A^2_{x,j})R(t_{x,j}, t_i) + \sum_{j=1}^{n}(A^3_j + A^4_j)R(t_j, t_i) \triangleq \Phi_{x,h}(t_i).$$

We then obtain the following linear system,

$$\begin{cases} R(t_1, t_1)m_{x,h}(t_1) + R(t_n, t_1)m_{x,h}(t_1) = \Phi_{x,h}(t_1). \\ \\ R(t_1, t_n)m_{x,h}(t_1) + R(t_n, t_n)m_{x,h}(t_n) = \Phi_{x,h}(t_n). \end{cases} \tag{42}$$

Solving (42) for $m_{x,h}(t_1)$ and $m_{x,h}(t_n)$ we obtain,

$$m_{x,h}(t_1) = \frac{R(t_n, t_n)\Phi_{x,h}(t_1) - R(t_1, t_n)\Phi_{x,h}(t_n)}{R(t_1, t_1)R(t_n, t_n) - R(t_1, t_n)^2}. \tag{43}$$

$$m_{x,h}(t_n) = \frac{R(t_1, t_1)\Phi_{x,h}(t_n) - R(t_1, t_n)\Phi_{x,h}(t_1)}{R(t_1, t_1)R(t_n, t_n) - R(t_1, t_n)^2}. \tag{44}$$

Finally, simple calculations yield,

$$m_{x,h}(t_1) = O\big(N_{T_n}\alpha_{n,h} + n\beta_{n,h}\big) \quad \text{and} \quad m_{x,h}(t_n) = O\big(N_{T_n}\alpha_{n,h} + n\beta_{n,h}\big).$$

This completes the proof of Lemma 3. □

6.5 Proof of Proposition 3.

Recall that $N_{T_n} = \text{Card } I_{x,h} = \text{Card }\{i = 1, \cdots, n \ : \ [t_{i-1}, t_{i+1}] \cap]x-h, x+h[\neq \emptyset\}$ and denote by $t_{x,i}$ the points of T_n for which $i \in I_{x,h}$, that is $T_n \cap]x-h, x+h[= \{t_{x,2}, \cdots, t_{x,N_{T_n}-1}\}$. Since $\mathbb{E}(\overline{Y}(t_i)) = g(t_i)$ then,

$$\begin{aligned}\mathbb{E}(\hat{g}^{pro}_n(x)) &= \sum_{j=1}^{n} m_{x,h}(t_j)g(t_j) \\ &= \sum_{i=1}^{N_{T_n}} m_{x,h}(t_{x,i})g(t_{x,i}) + \sum_{j=2}^{n-1} 1_{\{i \notin I_{x,h}\}} m_{x,h}(t_j)g(t_j) + m_{x,h}(t_1)g(t_1) + m_{x,h}(t_n)g(t_n).\end{aligned}$$

Using the asymptotic approximation of $m_{x,h|T_n}$ given in Lemma 3 we obtain,

$$E(\hat{g}^{pro}_n(x)) = \frac{1}{2}\sum_{i=1}^{N_{T_n}}(t_{x,i+1} - t_{x,i-1})\varphi_{x,h}(t_{x,i})g(t_{x,i}) + O\big(N_{T_n}\alpha_{n,h} + n\beta_{n,h}\big), \tag{45}$$

For $x \in [0, 1]$ let,

$$I_h(x) = \int_{x-h}^{x+h} \varphi_{x,h}(t)g(t)\, dt = \frac{1}{2}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i+1}} \varphi_{x,h}(t)g(t)\, dt,$$

and write,

$$\mathbb{E}(\hat{g}_n^{pro}(x)) = \mathbb{E}(\hat{g}_n^{pro}(x)) - I_h(x) + I_h(x) = \Delta_{x,h} + I_h(x) + O\Big(N_{T_n}\alpha_{n,h} + n\beta_{n,h}\Big), \qquad (46)$$

where,

$$\Delta_{x,h} = \frac{1}{2}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i+1}} \Big(\varphi_{x,h}(t_{x,i})g(t_{x,i}) - \varphi_{x,h}(t)g(t)\Big)\ dt.$$

We first control $\Delta_{x,h}$. We have,

$$\Delta_{x,h} = \frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} (g(t_{x,i}) - g(t))\ dt + \frac{1}{2}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i+1}} g(t)(\varphi_{x,h}(t_{x,i}) - \varphi_{x,h}(t))\ dt.$$

Since $\varphi_{x,h}$ is in C^1 and g is in C^2 then Taylor expansions of $\varphi_{x,h}$ and g give,

$$g(t) = g(t_{x,i}) + (t - t_{x,i})g'(t_{x,i}) + \frac{1}{2}(t - t_{x,i})^2 g''(\theta_{x,i}),$$

and,

$$\varphi_{x,h}(t) = \varphi_{x,h}(t_{x,i}) + (t - t_{x,i})\varphi'_{x,h}(\eta_{x,i}),$$

for some $\theta_{x,i}$ and $\eta_{x,i}$ between t and $t_{x,i}$. Thus,

$$\begin{aligned}
\Delta_{x,h} &= -\frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i})g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} (t - t_{x,i})\ dt - \frac{1}{4}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} g''(\theta_{x,i})(t - t_{x,i})^2\ dt \\
&- \frac{1}{2}\sum_{i=1}^{N_{T_n}} g(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} \varphi'_{x,h}(\eta_{x,i})(t - t_{x,i})\ dt - \frac{1}{2}\sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} \varphi'_{x,h}(\eta_{x,i})(t - t_{x,i})^2\ dt \\
&- \frac{1}{4}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i+1}} g''(\theta_{x,i})\varphi'_{x,h}(\eta_{x,i})(t - t_{x,i})^3\ dt.
\end{aligned}$$

Recall that g' and g'' are both bounded and that,

$$\sup_{0\le t\le 1} |\varphi_{x,h}(t)| < \frac{c}{h} \ \text{ and } \ \sup_{0\le t\le 1} |\varphi'_{x,h}(t)| < \frac{c'}{h^2}, \qquad (47)$$

for appropriate positive constants c and c'. Using this we obtain,

$$\begin{aligned}
\frac{1}{4}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} g''(\theta_{x,i})(t - t_{x,i})^2\ dt &= O\Big(\frac{N_{T_n}}{h} \sup_{0\le j\le 1} d_{j,n}^3\Big) \\
\frac{1}{2}\sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} \varphi'_{x,h}(\eta_{x,i})(t - t_{x,i})^2\ dt &= O\Big(\frac{N_{T_n}}{h^2} \sup_{0\le j\le 1} d_{j,n}^3\Big) \\
\frac{1}{4}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i+1}} g''(\theta_{x,i})\varphi'_{x,h}(\eta_{x,i})(t - t_{x,i})^2\ dt &= O\Big(\frac{N_{T_n}}{h^2} \sup_{0\le j\le 1} d_{j,n}^3\Big).
\end{aligned}$$

Thus,

$$\Delta_{x,h} = -\frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i})g'(t_{x,i})\int_{t_{x,i-1}}^{t_{x,i+1}}(t-t_{x,i})dt - \frac{1}{2}\sum_{i=1}^{N_{T_n}} g(t_{x,i})\varphi'_{x,h}(t_{x,i})\int_{t_{x,i-1}}^{t_{x,i+1}}(t-t_{x,i})\ dt$$
$$-\frac{1}{2}\sum_{i=1}^{N_{T_n}} g(t_{x,i})\int_{t_{x,i-1}}^{t_{x,i+1}}(t-t_{x,i})\Big(\varphi'_{x,h}(\eta_{x,i})-\varphi'_{x,h}(t_{x,i})\Big)\ dt + O\bigg(\frac{N_{T_n}}{h^2}\sup_{0\leq j\leq 1} d_{j,n}^3\bigg).$$

Since $\varphi'_{x,h}$ is Lipschitz then,

$$\sum_{i=1}^{N_{T_n}} g(t_{x,i})\int_{t_{x,i-1}}^{t_{x,i+1}}(t-t_{x,i})\Big(\varphi'_{x,h}(\eta_{x,i})-\varphi'_{x,h}(t_{x,i})\Big)\ dt = O\bigg(\frac{N_{T_n}}{h^3}\sup_{0\leq j\leq 1} d_{j,n}^3\bigg).$$

Thus,

$$\Delta_{x,h} = -\frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i})g'(t_{x,i})\int_{t_{x,i-1}}^{t_{x,i+1}}(t-t_{x,i})dt - \frac{1}{2}\sum_{i=1}^{N_{T_n}} g(t_{x,i})\varphi'_{x,h}(t_{x,i})\int_{t_{x,i-1}}^{t_{x,i+1}}(t-t_{x,i})\ dt$$
$$+ O\bigg(\frac{N_{T_n}}{h^3}\sup_{0\leq j\leq 1} d_{j,n}^3\bigg).$$

Basic integration gives,

$$\Delta_{x,h} = -\frac{1}{4}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i})g'(t_{x,i})(d_{x,i}^2-d_{x,i-1}^2) - \frac{1}{4}\sum_{i=1}^{N_{T_n}} g(t_{x,i})\varphi'_{x,h}(t_{x,i})(d_{x,i}^2-d_{x,i-1}^2)$$
$$+ O\bigg(\frac{N_{T_n}}{h^3}\sup_{0\leq j\leq 1} d_{j,n}^3\bigg).$$

We shall show that,

$$A \triangleq \sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i})g'(t_{x,i})(d_{x,i}^2-d_{x,i-1}^2) = O\bigg(\frac{N_{T_n}}{h^2}\sup_{0\leq j\leq 1} d_{j,n}^3\bigg),$$
$$B \triangleq \sum_{i=1}^{N_{T_n}} g(t_{x,i})\varphi'_{x,h}(t_{x,i})(d_{x,i}^2-d_{x,i-1}^2) = O\bigg(\frac{N_{T_n}}{h^3}\sup_{0\leq j\leq 1} d_{j,n}^3\bigg).$$

Starting with the term A. Recall that, since φ is of support $[x-h,x+h]$ and $t_{x,1}, t_{x,N_{T_n}-1} \notin]x-h,x+h[$, then $\varphi_{x,h}(t_{x,N_{T_n}}) = \varphi_{x,h}(t_{x,1}) = 0$ thus,

$$A = \sum_{i=2}^{N_{T_n}-1} \varphi_{x,h}(t_{x,i})g'(t_{x,i})d_{x,i}^2 - \sum_{i=1}^{N_{T_n}-2} \varphi_{x,h}(t_{x,i+1})g'(t_{x,i+1})d_{x,i}^2$$
$$= \sum_{i=2}^{N_{T_n}-2} \big(\varphi_{x,h}(t_{x,i})g'(t_{x,i}) - \varphi_{x,h}(t_{x,i+1})g'(t_{x,i+1})\big)d_{x,i}^2 + \bigg(\varphi_{x,h}(t_{x,N_{T_n}-1})g'(t_{x,N_{T_n}-1})d_{x,N_{T_n}-1}^2$$
$$- \varphi_{x,h}(t_{x,2})g'(t_{x,2})d_{x,1}^2\bigg)$$
$$\triangleq A_1 + A_2.$$

On the one hand, Taylor expansions of $\varphi_{x,h}$ around $t_{x,N_{T_n}}$ and $t_{x,1}$ yield,

$$\begin{aligned}
&\varphi_{x,h}(t_{x,N_{T_n}-1}) = (t_{x,N_{T_n}-1} - t_{x,N_{T_n}})\varphi'_{x,h}(\gamma_{x,N_{T_n}}),\\
&\varphi_{x,h}(t_{x,2}) = (t_{x,2} - t_{x,1})\varphi'_{x,h}(\gamma_{x,1}),
\end{aligned}$$

for some $\gamma_{x,N_{T_n}} \in]t_{x,N_{T_n}-1}, t_{x,N_{T_n}}[$ and some $\gamma_{x,1} \in]t_{x,1}, t_{x,2}[$. Using (47) and the fact that g' is bounded we obtain,

$$A_2 = O\Big(\frac{1}{h^2} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

On the other hand we have,

$$\begin{aligned}
A_1 &= \sum_{i=2}^{N_{T_n}-2} \big(\varphi_{x,h}(t_{x,i})g'(t_{x,i}) - \varphi_{x,h}(t_{x,i+1})g'(t_{x,i+1})\big)d_{x,i}^2\\
&= \sum_{i=2}^{N_{T_n}-2} \varphi_{x,h}(t_{x,i})\big(g'(t_{x,i}) - g'(t_{x,i+1})\big)d_{x,i}^2 + \sum_{i=2}^{N_{T_n}-2} g'(t_{x,i+1})\big(\varphi_{x,h}(t_{x,i}) - \varphi_{x,h}(t_{x,i+1})\big)d_{x,i}^2.
\end{aligned}$$

Since $\varphi_{x,h}$ is in C^1 and g is in C^2 then using (47), we obtain,

$$A_1 = O\Big(\frac{N_{T_n}}{h^2} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

In a similar way and from Assumption (D), we obtain,

$$B = O\Big(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

Hence,

$$\Delta_{x,h} = O\Big(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

Thus using (46),

$$\mathbb{E}(\hat{g}_n^{pro}(x)) = I_h(x) + O(N_{T_n}\alpha_{n,h} + n\beta_{n,h}) + O\Big(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

The control of $I_h(x)$ is classical and it can bee seen from [16] that,

$$I_h(x) = g(x) + \frac{1}{2}h^2 g''(x)\int_{-1}^{1} t^2 K(t)\, dt + o(h^2). \tag{48}$$

Finally,

$$\mathbb{E}(\hat{g}_n^{pro}(x)) - g(x) = \frac{1}{2}h^2 g''(x)\int_{-1}^{1} t^2 K(t)\, dt + o(h^2) + O\Big(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3 + N_{T_n}\alpha_{n,h} + n\beta_{n,h}\Big).$$

This concludes the proof of Proposition 3. □

6.6 Proof of Proposition 4.

Let $t_0 = 0,\ t_{n+1} = 1$ and set $\overline{Y}(t_0) = 0$ and $\overline{Y}(t_{n+1}) = \overline{Y}(t_n)$. Recall that,

$$\hat{g}_n^{pro}(x) = \sum_{i=1}^{n+1} \overline{Y}(t_i) \int_{t_{i-1}}^{t_i} \varphi_{x,h}(s)ds + \sum_{i=0}^{n} \frac{\overline{Y}(t_{i+1}) - \overline{Y}(t_i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} (s - t_{i+1})\varphi_{x,h}(s)ds.$$

Since $\mathbb{E}\ (\overline{Y}(t_i)) = g(t_i)$ then,

$$\mathbb{E}\ (\hat{g}_n^{pro}(x)) = \sum_{i=1}^{n+1} g(t_i) \int_{t_{i-1}}^{t_i} \varphi_{x,h}(s)ds + \sum_{i=0}^{n} \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} (s - t_{i+1})\varphi_{x,h}(s)ds.$$

Recall that $N_{T_n} = \text{Card } I_{x,h} = \{i = 1, \cdots, n\ :\ [t_{i-1,n}, t_{i+1,n}] \cap]x - h, x + h[\neq \emptyset\}$ and denote by $t_{x,i}$ the points of T_n for which $i \in I_{x,h}$. Using the support of $\varphi_{x,h}$ we obtain,

$$\mathbb{E}\ (\hat{g}_n^{pro}(x)) = \sum_{i=1}^{N_{T_n}} g(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s)ds + \sum_{i=1}^{N_{T_n}} \frac{g(t_{x,i+1}) - g(t_{x,i})}{t_{x,i+1} - t_{x,i}} \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})\varphi_{x,h}(s)ds.$$

Let $d_{x,i} = t_{x,i+1} - t_{x,i}$. Since g is in C^2 and $\varphi_{x,h}$ is in C^1 then Taylor expansions of g around $t_{x,i}$ and of $\varphi_{x,h}$ around $t_{x,i+1}$ yield,

$$\begin{aligned} g(t_{x,i+1}) &= g(t_{x,i}) + d_{x,i}\ g'(t_{x,i}) + \frac{1}{2} d_{x,i}^2\ g''(\theta_{x,i}), \\ \varphi_{x,h}(s) &= \varphi_{x,h}(t_{x,i+1}) + (s - t_{x,i+1})\varphi'_{x,h}(s_i). \end{aligned}$$

for some $\theta_{x,i} \in]t_{x,i}, t_{x,i+1}[$ and some $s_i \in]s, t_{x,i+1}[$. Recall that, using the support of φ, $\varphi_{x,h}(t_{x,1}) = \varphi_{x,h}(t_{x,N_{T_n}}) = 0$ thus,

$$\begin{aligned} \mathbb{E}\ (\hat{g}_n^{pro}(x)) &= \sum_{i=1}^{N_{T_n}} g(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s)ds\ + \sum_{i=1}^{N_{T_n}-2} g'(t_{x,i})\varphi_{x,h}(t_{x,i+1}) \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})ds \\ &+ \sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})^2 \varphi'_{x,h}(s_i)ds + \frac{1}{2} \sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i+1}) g''(\theta_{x,i}) d_{x,i} \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})ds \\ &+ \frac{1}{2} \sum_{i=1}^{N_{T_n}} g''(\theta_{x,i}) d_{x,i} \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})^2 \varphi'_{x,h}(s_i)ds. \end{aligned}$$

Recall that g' and g'' are bounded, Lemma 2 yields $N_{T_n} = O(nh)$ and $d_{x,i} = O(\frac{1}{n})$ and using (47) we obtain,

$$\begin{aligned} &\sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})^2 \varphi'_{x,h}(s_i)ds = O\left(\frac{1}{n^2 h}\right). \\ &\frac{1}{2} \sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i+1}) g''(\theta_{x,i}) d_{x,i} \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})ds = O\left(\frac{1}{n^2}\right). \\ &\frac{1}{2} \sum_{i=1}^{N_{T_n}} g''(\theta_{x,i}) d_{x,i} \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})^2 \varphi'_{x,h}(s_i)ds = O\left(\frac{1}{n^3 h}\right). \end{aligned}$$

It follows that by simple integration,

$$
\begin{aligned}
\mathbb{E}\ (\hat{g}_n^{pro}(x)) &= \sum_{i=1}^{N_{T_n}} g(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s) ds - \frac{1}{2} \sum_{i=1}^{N_{T_n}-2} g'(t_{x,i}) \varphi_{x,h}(t_{x,i+1}) d_{x,i}^2 + O\left(\frac{1}{n^2 h}\right) \\
&= \sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s) g(s)\ ds + \sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s)(g(t_{x,i}) - g(s))\ ds \\
&\quad - \frac{1}{2} \sum_{i=1}^{N_{T_n}-2} g'(t_{x,i}) \varphi_{x,h}(t_{x,i+1}) d_{x,i}^2 + O\left(\frac{1}{n^2 h}\right).
\end{aligned}
$$

On the one hand, we have,

$$
\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s) g(s)\ ds = \int_{x-h}^{x+h} \varphi_{x,h}(s) g(s)\ ds.
$$

On the other hand, Taylor expansion of g and $\varphi_{x,h}$ arround $t_{x,i}$ yield,

$$
\begin{aligned}
g(t_{x,i}) &= g(s) + (t_{x,i} - s) g'(t_{x,i}) - \frac{1}{2}(t_{x,i} - s)^2 g''(s_i'), \\
\varphi_{x,h}(s) &= \varphi_{x,h}(t_{x,i}) + (s - t_{x,i}) \varphi_{x,h}'(s_i'').
\end{aligned}
$$

for some s_i' and s_i'' in $]s, t_{x,i}[$. Thus,

$$
\begin{aligned}
\mathbb{E}\ (\hat{g}_n^{pro}(x)) &= \int_{x-h}^{x+h} \varphi_{x,h}(s) g(s)\ ds + \sum_{i=2}^{N_{T_n}-1} g'(t_{x,i}) \varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} (t_{x,i} - s)\ ds \\
&- \sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} (t_{x,i} - s)^2 \varphi_{x,h}'(s_i'')\ ds - \frac{1}{2} \sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} g''(s_i')(t_{x,i} - s)^2\ ds \\
&+ \frac{1}{2} \sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i}} g''(s_i') \varphi_{x,h}'(s_i'')(t_{x,i} - s)^3\ ds - \frac{1}{2} \sum_{i=1}^{N_{T_n}-2} g'(t_{x,i}) \varphi_{x,h}(t_{x,i+1}) d_{x,i}^2 \\
&+ O\left(\frac{1}{n^2 h}\right).
\end{aligned}
$$

Using the boundedness of g' and g'' in addition to Lemma 2 and Equation (47), we obtain,

$$
\begin{aligned}
&\sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} (t_{x,i} - s)^2 \varphi_{x,h}'(s_i'')\ ds = O\left(\frac{1}{n^2 h}\right). \\
&\frac{1}{2} \sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} g''(s_i')(t_{x,i} - s)^2\ ds = O\left(\frac{1}{n^2}\right). \\
&\frac{1}{2} \sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i}} g''(s_i') \varphi_{x,h}'(s_i'')(t_{x,i} - s)^3\ ds = O\left(\frac{1}{n^3 h}\right).
\end{aligned}
$$

Thus,

$$\mathbb{E}\ (\hat{g}_n^{pro}(x)) = \int_{x-h}^{x+h} \varphi_{x,h}(s) g(s)\ ds + \frac{1}{2}\sum_{i=2}^{N_{T_n}-2} g'(t_{x,i})\varphi_{x,h}(t_{x,i}) d_{x,i-1}^2$$
$$- \frac{1}{2}\sum_{i=1}^{N_{T_n}-2} g'(t_{x,i})\varphi_{x,h}(t_{x,i+1}) d_{x,i}^2 + O\Big(\frac{1}{n^2 h}\Big)$$
$$= \int_{x-h}^{x+h} \varphi_{x,h}(s) g(s)\ ds + \frac{1}{2}\sum_{i=1}^{N_{T_n}-2} \Big(g'(t_{x,i+1}) - g'(t_{x,i})\Big)\varphi_{x,h}(t_{x,i+1}) d_{x,i}^2 + O\Big(\frac{1}{n^2 h}\Big).$$

Since g' is Lipschitz, then we have,

$$\mathbb{E}\ (\hat{g}_n^{pro}(x)) = \int_{x-h}^{x+h} \varphi_{x,h}(s)\ g(s)\ ds + O\Big(\frac{1}{n^2 h}\Big). \tag{49}$$

Finally, from (48) we obtain,

$$\mathbb{E}\ (\hat{g}_n^{pro}(x)) - g(x) = \frac{1}{2} h^2 g''(x) \int_{-1}^{1} t^2 K(t) dt + o(h^2) + O\Big(\frac{1}{n^2 h}\Big).$$

This concludes the proof of Proposition 4. $\square$

6.7 Proof of Proposition 5.

The great lines of this proof are based on Sacks and Ylvisaker (1966) [25]. From the definition of the orthogonal projection (see the Appendix) and using the Pythagore theorem we obtain,

$$m\Big(\frac{\sigma_{x,h}^2}{m} - \mathrm{Var} g_n^{pro}(x)\Big) = ||f_{x,h}||^2 - ||P_{|T_n} f_{x,h}||^2 = ||f_{x,h} - P_{|T_n} f_{x,h}||^2, \tag{50}$$

where $P_{|T_n} f_{x,h}$ is the orthogonal projection of $f_{x,h}$ on the subspace of $\mathcal{F}(\varepsilon)$ spanned by $\{R(\cdot, t_i), t_i \in T_n\}$, denoted here by V_{T_n}. We shall then prove that,

$$||f_{x,h} - P_{|T_n} f_{x,h}||^2 \leq \frac{C}{h} \sup_{0\leq j\leq n} d_{j,n}^2. \tag{51}$$

Recall that $N_{T_n} = \mathrm{Card}\ I_{x,h} = \mathrm{Card}\ I_{x,h} = \{i = 1, \cdots, n\ :\ [t_{i-1,n}, t_{i+1,n}] \cap]x-h, x+h[\neq \emptyset\}$ and denote by $t_{x,i}$ the points of T_n for which $i \in I_{x,h}$. Let $g_n := g_{n,x} = \sum_{i=1}^n \gamma_{x,i} R(\cdot, t_{x,i})$ with $\gamma_{x,i} = 0$ for every $i \notin I_{x,h}$. It is clear that $g_n \in V_{T_n}$ and thus from the definition of the orthogonal projection we have,

$$||f_{x,h} - P_{|T_n} f_{x,h}||^2 \leq ||f_{x,h} - g_n||^2.$$

Now using (F1) in the Appendix and the support of $\varphi_{x,h}$ we obtain,

$$||f_{x,h} - g_n||^2 = \int_0^1 (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t)\ dt - \sum_{i=1}^n (f_{x,h}(t_i) - g_n(t_i))\gamma_{x,i}$$
$$= \int_{x-h}^{x+h} (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t)\ dt - \sum_{i=1}^{N_{T_n}} (f_{x,h}(t_{x,i}) - g_n(t_{x,i}))\gamma_{x,i} \tag{52}$$

In what follows, we distinguish between three cases according to the location of $t_{x,1}$ and $t_{x,N_{T_n}}$ in the interval $[x-h, x+h]$.

First case. Suppose first that $t_{x,1} = x-h$ and $t_{x,N_{T_n}} = x+h$ and take,

$$\gamma_{x,i} = \int_{t_{x,i}}^{t_{x,i+1}} \varphi_{x,h}(t)\, dt \quad \text{for} \quad i = 1, \ldots, N_{T_n} - 1. \tag{53}$$

we have in this case,

$$||f_{x,h} - g_n||^2 = \sum_{i=1}^{N_{T_n}} \int_{t_{x,i}}^{t_{x,i+1}} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,i}) - g_n(t_{x,i})) \Big) \varphi_{x,h}(t)\, dt. \tag{54}$$

Assumption (A) yields that $f_{x,h}$ is twice differentiable on $[0,1]$, while g_n is twice differentiable everywhere except on T_n, but it has left and right derivatives. Taylor expansion of $f_{x,h} - g_n$ around $t_{x,i}$ for $i = 1, \cdots, N_{T_n} - 1$ and $t \in]t_{x,i}, t_{x,i+1}[$ gives,

$$\begin{aligned} f_{x,h}(t) - g_n(t) &= (f_{x,h}(t_{x,i}) - g_n(t_{x,i})) + (t - t_{x,i})(f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i})) \\ &\quad + \frac{1}{2}(t - t_{x,i})^2 (f''_{x,h}(\theta_{x,t}) - g''_n(\theta^+_{x,t})), \end{aligned} \tag{55}$$

for some $\theta_{x,t} \in]t_{x,i}, t[$. On the one hand, we have,

$$g'_n(t^+_{x,i}) = \sum_{j=1}^{N_{T_n}-1} R^{(0,1)}(t_{x,j}, t^+_{x,i}) \gamma_{x,j}. \tag{56}$$

On the other hand, using (32) we obtain,

$$\begin{aligned} f'_{x,h}(t_{x,i}) &= \int_{x-h}^{x+h} R^{(0,1)}(s, t^+_{x,i}) \varphi_{x,h}(s)\, ds = \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} R^{(0,1)}(s, t^+_{x,i}) \varphi_{x,h}(s)\, ds \\ &= \sum_{\substack{j=1 \\ j \neq i}}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} R^{(0,1)}(s, t^+_{x,i}) \varphi_{x,h}(s)\, ds \; + \int_{t_{x,i}}^{t_{x,i+1}} R^{(0,1)}(s, t^+_{x,i}) \varphi_{x,h}(s)\, ds. \end{aligned} \tag{57}$$

When $j \neq i$ we have,

$$\int_{t_{x,j}}^{t_{x,j+1}} R^{(0,1)}(s, t^+_{x,i}) \varphi_{x,h}(s)\, ds = R^{(0,1)}(t_{x,j}, t_{x,i}) \gamma_{x,j} + \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j}) R^{(1,1)}(\delta_{s,j}, t_{x,i}) \varphi_{x,h}(s)\, ds, \tag{58}$$

for some $\delta_{s,j} \in]t_{x,j}, s[$, while for $j = i$ we have,

$$\begin{aligned} \int_{t_{x,i}}^{t_{x,i+1}} R^{(0,1)}(s, t^+_{x,i}) \varphi_{x,h}(s)\, ds &= \int_{t_{x,i}}^{t_{x,i+1}} R^{(0,1)}(s, t^-_{x,i}) \varphi_{x,h}(s)\, ds \\ &= R^{(0,1)}(t_{x,i}, t^-_{x,i}) \gamma_{x,i} + \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i}) R^{(1,1)}(\delta^+_{s,i}, t^-_{x,i}) \varphi_{x,h}(s)\, ds. \end{aligned} \tag{59}$$

Collecting (56), (57), (58) and (59) we obtain,

$$
\begin{aligned}
f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i}) &= \sum_{\substack{j=1\\ j\neq i}}^{N_{T_n}-1} R^{(0,1)}(t_{x,j}, t_{x,i})\gamma_{x,j} + \sum_{\substack{j=1\\ j\neq i}}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j}) R^{(1,1)}(\delta_{s,j}, t_{x,i})\varphi_{x,h}(s)\, ds \\
&+ R^{(0,1)}(t_{x,i}, t^-_{x,i})\gamma_{x,i} + \int_{t_{x,i}}^{t_{x,i+1}} R^{(1,1)}(\delta^+_{s,i}, t^-_{x,i})\varphi_{x,h}(s)\, ds - \sum_{j=1}^{N_{T_n}-1} R^{(0,1)}(t_{x,j}, t^+_{x,i})\gamma_{x,j} \\
&= \alpha(t_{x,i})\gamma_{x,i} + \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j}) R^{(1,1)}(\delta^+_{s,j}, t^-_{x,i})\varphi_{x,h}(s)\, ds.
\end{aligned}
$$

It is easy to see that,

$$
\begin{aligned}
|f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i})| &\leq \alpha_1\gamma_{x,i} + \frac{K_\infty}{h} R_1 \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j})\, ds \\
&\leq \frac{K_\infty}{h}\alpha_1 d_{x,i} + \frac{K_\infty}{2h} R_1 \sum_{j=1}^{N_{T_n}-1} d^2_{x,j}. \qquad (60)
\end{aligned}
$$

We deduce from (33) that for all $\theta_{x,t} \in]t_{x,i}, t_{x,i+1}[$ we have,

$$
|f''_{x,h}(\theta_{x,t})| \leq \frac{K_\infty}{h}\alpha_1 + \frac{K_\infty}{h} R_2 \times 2h = \frac{K_\infty}{h}\alpha_1 + 2K_\infty R_2.
$$

In addition, for $\theta_{x,t} \in]t_{x,i}, t_{x,i+1}[$ we have,

$$
|g''_n(\theta^+_{x,t})| = \left| \sum_{j=1}^{N_{T_n}-1} R^{(0,2)}(t_{x,j}, \theta^+_{x,t})\gamma_{x,j} \right| \leq \frac{K_\infty}{h} R_2 \sum_{j=1}^{N_{T_n}-1} d_{x,j} = \frac{K_\infty}{h} R_2 \times 2h = 2K_\infty R_2,
$$

Thus,

$$
|f''_{x,h}(\theta_{x,t}) - g''_n(\theta^+_{x,t})| \leq \frac{K_\infty}{h}\alpha_1 + 4K_\infty R_2. \qquad (61)
$$

Equations (55), (60) and (61) yield that for $i = 1, \cdots, N_{T_n} - 1$,

$$
\begin{aligned}
&\left| \int_{t_{x,i}}^{t_{x,i+1}} \left[(f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,i}) - g_n(t_{x,i}))\right] \varphi_{x,h}(t)\, dt \right| \\
&\leq \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i}) |f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i})| |\varphi_{x,h}(t)|\, dt \\
&\quad + \frac{1}{2} \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 |f''_{x,h}(\theta_{x,t}) - g''_n(\theta^+_{x,t})| |\varphi_{x,h}(t)|\, dt \\
&\leq \left(\frac{K_\infty}{h}\alpha_1 d_{x,i} + \frac{K_\infty}{2h} R_1 \sum_{j=1}^{N_{T_n}-1} d^2_{x,j} \right) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i}) |\varphi_{x,h}(t)|\, dt \\
&\quad + \frac{1}{2} \left(\frac{K_\infty}{h}\alpha_1 + 4K_\infty R_2 \right) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 |\varphi_{x,h}(t)|\, dt \\
&\leq \left(\frac{K_\infty}{h}\alpha_1 d_{x,i} + \frac{K_\infty}{2h} R_1 \sum_{j=1}^{N_{T_n}-1} d^2_{x,j} \right) \frac{K_\infty}{2h} d^2_{x,i} + \frac{1}{2} \left(\frac{K_\infty}{h}\alpha_1 + 4K_\infty R_2 \right) \frac{K_\infty}{3h} d^3_{x,i} \\
&\leq \frac{K^2_\infty}{4h^2} R_1 d^2_{x,i} \sum_{j=1}^{N_{T_n}-1} d^2_{x,j} + \frac{2K^2_\infty}{3h} \left(\frac{\alpha_1}{h} + R_2 \right) d^3_{x,i}. \qquad (62)
\end{aligned}
$$

Injecting this inequality in (54) yields,

$$
\begin{aligned}
||f_{x,h} - P_{|T_n} f_{x,h}||^2 &\leq \frac{K_\infty^2}{4h^2} R_1 \Bigg(\sum_{i=1}^{N_{T_n}-1} d_{x,i}^2 \Bigg)^2 + \frac{2K_\infty^2}{3h} \Big(\frac{\alpha_1}{h} + R_2 \Big) \sum_{i=1}^{N_{T_n}-1} d_{x,i}^3 \\
&\leq \frac{K_\infty^2}{4h^2} R_1 \sup_{1\leq i\leq n} d_{i,n}^2 \Bigg(\sum_{i=1}^{N_{T_n}-1} d_{x,i} \Bigg)^2 + \frac{2K_\infty^2}{3h} \big(\frac{\alpha_1}{h} + R_2 \big) \sup_{1\leq i\leq n} d_{i,n}^2 \sum_{i=1}^{N_{T_n}-1} d_{x,i}.
\end{aligned}
$$

Since $\sum_{i=1}^{N_{T_n}-1} d_{x,i} = 2h$ then,

$$
||f_{x,h} - P_{|T_n} f_{x,h}||^2 \leq \left(\frac{4}{3h}\alpha_1 + R_1 + \frac{4}{3} R_2 \right) K_\infty^2 \sup_{1\leq i\leq n} d_{i,n}^2
$$

Finally, since $h < 1$ then,

$$
||f_{x,h} - P_{|T_n} f_{x,h}||^2 \leq \left(\frac{4}{3}\alpha_1 + R_1 + \frac{4}{3} R_2 \right) K_\infty^2 \frac{1}{h} \sup_{1\leq i\leq n} d_{i,n}^2.
$$

Proposition 5 is then proved for the first case.

Second case. Consider now the case where $t_{x,1} < x - h$ and $t_{x,N_{T_n}} > x + h$. For $i = 2, \ldots, N_{T_n} - 2$ set,

$$
\gamma_{x,i} = \int_{t_{x,i}}^{t_{x,i+1}} \varphi_{x,h}(t)\, dt,\ \gamma_{x,1} = \int_{x-h}^{t_{x,2}} \varphi_{x,h}(t)\, dt,\ \gamma_{x,N_{T_n}-1} = \int_{t_{x,N_{T_n}-1}}^{x+h} \varphi_{x,h}(t)\, dt \text{ and } \gamma_{x,N_{T_n}} = 0. \tag{63}
$$

Using this we obtain,

$$
\begin{aligned}
||f_{x,h} - g_n||^2 &= \int_{x-h}^{t_{x,2}} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,1}) - g_n(t_{x,1})) \Big) \varphi_{x,h}(t)\, dt \\
&\quad + \sum_{i=2}^{N_{T_n}} \int_{t_{x,i}}^{t_{x,i+1}} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,i}) - g_n(t_{x,i})) \Big) \varphi_{x,h}(t)\, dt \\
&\quad + \int_{t_{x,N_{T_n}}}^{x+h} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,N_{T_n}}) - g_n(t_{x,N_{T_n}})) \Big) \varphi_{x,h}(t)\, dt.
\end{aligned} \tag{64}
$$

We first control the first term of (64). Let,

$$
A_{x,h}^{(1)} = \int_{x-h}^{t_{x,2}} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,1}) - g_n(t_{x,1})) \Big) \varphi_{x,h}(t)\, dt.
$$

For $t \in]x - h, t_{x,2}[$ we have,

$$
\begin{aligned}
f_{x,h}(t) - g_n(t) &= (f_{x,h}(t_{x,1}) - g_n(t_{x,1})) + (t - t_{x,1})(f'_{x,h}(t_{x,1}) - g'_n(t_{x,1}^+)) \\
&\quad + \frac{1}{2}(t - t_{x,1})^2 (f''_{x,h}(\theta_{x,1}) - g''_n(\theta_{x,1}^+)),
\end{aligned} \tag{65}
$$

for some $\theta_{x,1} \in]x-h, t[$. Equation (32) yields,

$$f'_{x,h}(t_{x,1}) = \int_{x-h}^{x+h} R^{(0,1)}(s, t^+_{x,1})\varphi_{x,h}(s)\, ds = \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} R^{(0,1)}(s, t^+_{x,1})\varphi_{x,h}(s)\, ds$$

$$= \int_{x-h}^{t_{x,2}} R^{(0,1)}(s, t^-_{x,1})\varphi_{x,h}(s)\, ds \; + \sum_{j=2}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} R^{(0,1)}(s, t^+_{x,1})\varphi_{x,h}(s)\, ds$$

$$= R^{(0,1)}(t_{x,1}, t^-_{x,1})\gamma_{x,1} + \int_{x-h}^{t_{x,2}} (s - t_{x,1}) R^{(1,1)}(\delta^+_{s,1}, t^-_{x,1})\varphi_{x,h}(s)\, ds$$

$$+ \sum_{j=2}^{N_{T_n}-1} R^{(0,1)}(t_{x,j}, t_{x,1})\gamma_{x,j} \; + \sum_{j=2}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j}) R^{(1,1)}(\delta_{s,j}, t^+_{x,1})\varphi_{x,h}(s)\, ds. \quad (66)$$

Recall that,

$$g'_n(t^+_{x,1}) = R^{(0,1)}(t_{x,1}, t^+_{x,1})\gamma_{x,1} + \sum_{j=2}^{N_{T_n}-1} R^{(0,1)}(t_{x,j}, t_{x,1})\gamma_{x,j}. \quad (67)$$

Equations (66) and (67) give,

$$f'_{x,h}(t_{x,1}) - g'_n(t^+_{x,1}) = \alpha(t_{x,1})\gamma_{x,1} + \sum_{j=2}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j}) R^{(1,1)}(\delta_{s,j}, t^+_{x,1})\varphi_{x,h}(s)\, ds$$
$$+ \int_{x-h}^{t_{x,2}} (s - t_{x,1}) R^{(1,1)}(\delta^+_{s,1}, t^-_{x,1})\varphi_{x,h}(s)\, ds.$$

Note that $t_{x,2} - (x-h) \le \sup_{1\le i\le n} d_{i,n}$. We obtain,

$$|f'_{x,h}(t_{x,1}) - g'_n(t^-_{x,1})| \le \frac{K_\infty}{h}\alpha_1 \sup_{1\le i\le n} d_{i,n} + \frac{K_\infty}{2h} R_1 \sum_{j=2}^{N_{T_n}-1} d^2_{x,j} + \frac{K_\infty}{2h} R_1 \sup_{1\le i\le n} d^2_{i,n}$$
$$\le \frac{K_\infty}{h}\alpha_1 \sup_{1\le i\le n} d_{i,n} + K_\infty R_1 \sup_{1\le i\le n} d_{i,n} + \frac{K_\infty}{2h} R_1 \sup_{1\le i\le n} d^2_{i,n}$$
$$\le K_\infty\Big(\frac{\alpha_1}{h} + \frac{3}{2}R_1\Big) \sup_{1\le i\le n} d_{i,n} \quad (68)$$

By (61) we have,

$$|f''_{x,h}(\theta_{x,t}) - g''_n(\theta^-_{x,t})| \le \frac{K_\infty}{h}\alpha_1 + 4K_\infty R_2. \quad (69)$$

Equations (65), (68) and (69) yield,

$$|A^{(1)}_{x,h}| \le \; |f'_{x,h}(t_{x,1}) - g'_n(t^+_{x,1})| \int_{x-h}^{t_{x,2}} (t - t_{x,1})|\varphi_{x,h}(t)|\, dt$$
$$+ \frac{1}{2}\int_{x-h}^{t_{x,2}} (t - t_{x,1})^2 |f''_{x,h}(\theta_{x,1}) - g''_n(\theta^+_{x,1})||\varphi_{x,h}(t)|\, dt$$
$$\le \Big(K_\infty\Big(\frac{\alpha_1}{h} + \frac{3}{2}R_1\Big)\sup_{1\le i\le n} d_{i,n}\Big)\frac{K_\infty}{2h}\sup_{1\le i\le n} d^2_{i,n} + \Big(\frac{K_\infty}{h}\alpha_1 + 4K_\infty R_2\Big)\frac{K_\infty}{6h}\sup_{1\le i\le n} d^3_{i,n}$$
$$\le \Big(\frac{2}{3}\alpha_1 + \frac{3}{4}R_1 + \frac{2}{3}R_2\Big)\frac{K^2_\infty}{h^2}\sup_{1\le i\le n} d^3_{i,n}. \quad (70)$$

Similarly we obtain,

$$A^{(2)}_{x,h} \triangleq \int_{t_{x,N_{T_n}}}^{x+h} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,N_{T_n}}) - g_n(t_{x,N_{T_n}})) \Big) \varphi_{x,h}(t)\, dt$$

$$|A^{(2)}_{x,h}| \leq \left(\frac{2}{3}\alpha_1 + \frac{3}{4}R_1 + \frac{2}{3}R_2 \right) \frac{K^2_\infty}{h^2} \sup_{1\leq i\leq n} d^3_{i,n}. \tag{71}$$

Thus,

$$|A^{(1)}_{x,h} + A^{(2)}_{x,h}| \leq \left(\frac{4}{3}\alpha_1 + \frac{3}{2}R_1 + \frac{4}{3}R_2 \right) \frac{K^2_\infty}{h^2} \sup_{1\leq i\leq n} d^3_{i,n}.$$

For $i = 2, \dots, N_{T_n} - 2$, similar calculations as those leading to (62) give,

$$\Big| \int_{t_{x,i}}^{t_{x,i+1}} \big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,i}) - g_n(t_{x,i})) \big) \varphi_{x,h}(t)\, dt \Big|$$
$$\leq \frac{K^2_\infty}{4h^2} R_1 d^2_{x,i} \sum_{j=1}^{N_{T_n}} d^2_{x,j} + \frac{2K^2_\infty}{3h} (\frac{\alpha_1}{h} + R_2) d^3_{x,i}.$$

Thus,

$$\Big| \sum_{i=2}^{N_{T_n}-2} \int_{t_{x,i}}^{t_{x,i+1}} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,i}) - g_n(t_{x,i})) \Big) \varphi_{x,h}(t)\, dt \Big|$$
$$\leq \left(\frac{4}{3}\alpha_1 + R_1 + \frac{4}{3}R_2 \right) \frac{K^2_\infty}{h} \sup_{1\leq i\leq n} d^2_{i,n}. \tag{72}$$

Then, Equations (70), (71) and (72) yield,

$$\begin{aligned} ||f_{x,h} - P_{T_n} f_{x,h}||^2 &\leq \left(\frac{4}{3}\alpha_1 + \frac{3}{2}R_1 + \frac{4}{3}R_2 \right) \frac{K^2_\infty}{h} \sup_{1\leq i\leq n} d^2_{i,n} + \left(\frac{4}{3}\alpha_1 + R_1 + \frac{4}{3}R_2 \right) \frac{K^2_\infty}{h^2} \sup_{1\leq i\leq n} d^3_{i,n} \\ &= \left(\frac{8}{3}\alpha_1 + \frac{5}{2}R_1 + \frac{8}{3}R_2 \right) \frac{K^2_\infty}{h} \sup_{1\leq i\leq n} d^2_{i,n} \end{aligned}$$

Third case. Suppose now that $t_{x,1} = x - h$ and $t_{x,N_{T_n}} > x + h$ (respectively $t_{x,1} < x - h$ and $t_{x,N_{T_n}} = x + h$). Let $T_{n-1} = T_n - \{x - h\}$ (respectively $T_{n-1} = T_n - \{x + h\}$). Since $P_{T_{n-1}} f_{x,h} \in V_{T_n}$ we obtain,

$$||f_{x,h} - P_{T_n} f_{x,h}||^2 \leq ||f_{x,h} - P_{T_{n-1}} f_{x,h}||^2,$$

we can then apply the result of the second case to the right side of the previous inequality. The proof of Proposition 5 is complete. □

6.8 Proof of Proposition 6.

The great lines of this proof are based on the work of [25]. Keeping Equation (50) in mind we deduce that Equation (9) is equivalent to,

$$\varliminf_{n\to\infty} \frac{N^2_{T_n}}{h} ||f_{x,h} - P_{|T_n} f_{x,h}||^2 \geq \frac{1}{12}\alpha(x) \Big\{ \int_{-1}^{1} K^{2/3}(t) dt \Big\}^3. \tag{73}$$

We shall take the same notation as in the previous proof. Let $g_n = P_{|T_n} f_{x,h}$, it is shown by Equation (106) in the Appendix that:

$$g_n(t_i) = f_{x,h}(t_i) = \sum_{j=1}^{n} R(t_j, t_i) m_{x,h}(t_j), \quad \text{for } i = 1, \cdots, n.$$

We have from (F1) in the Appendix that,

$$||f_{x,h} - g_n||^2 = \int_0^1 (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t)\, dt - \sum_{i=1}^{n} m_{x,h}(t_i)(f_{x,h}(t_i) - g_n(t_i))$$
$$= \int_{x-h}^{x+h} (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t) dt.$$

Suppose first that $t_{x,1} = x - h$ and $t_{x,N_{T_n}} = x + h$, then the last equalities give,

$$||f_{x,h} - g_n||^2 = \sum_{i=1}^{N_{T_n}-1} \int_{t_{x,i}}^{t_{x,i+1}} (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t) dt. \tag{74}$$

Under Assumptions (A) and (B), the function $f_{x,h}$ is twice differentiable at every $t \in [0,1]$ and g_n is twice differentiable at every $t \in [0,1]$ except on T_n, however, it has left and right derivatives. We expand $(f_{x,h} - g_n)$ in a Taylor series around $t_{x,i}$ for $t \in]t_{x,i}, t_{x,i+1}[$ up to order 2 we obtain,

$$f_{x,h}(t) - g_n(t) = (f_{x,h}(t_{x,i}) - g_n(t_{x,i})) + (t - t_{x,i})(f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i}))$$
$$+ \frac{1}{2}(t - t_{x,i})^2 (f''_{x,h}(\sigma_{x,t}) - g''_n(\sigma^+_{x,t})),$$

for some $\sigma_{x,t} \in]t_{x,i}, t[$. Since $g_n(t_{x,i}) = f_{x,h}(t_{x,i})$ then,

$$f_{x,h}(t) - g_n(t) = (t - t_{x,i})(f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i})) + \frac{1}{2}(t - t_{x,i})^2 (f''_{x,h}(\sigma_{x,t}) - g''_n(\sigma^+_{x,t})), \tag{75}$$

On the one hand, we have for $i \in 1, \ldots, N_{T_n} - 1$,

$$f_{x,h}(t_{x,i+1}) - g_n(t_{x,i+1}) = d_{x,i}(f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i})) + \frac{1}{2} d^2_{x,i}(f''_{x,h}(\sigma_{x,i}) - g''_n(\sigma^+_{x,i})).$$

for some $\sigma_{x,i} \in]t_{x,i}, t_{x,i+1}[$. Thus,

$$f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i}) = -\frac{1}{2} d_{x,i}(f''_{x,h}(\sigma_{x,i}) - g''_n(\sigma^+_{x,i})). \tag{76}$$

On the other hand, it is shown by (F4) in the Appendix that,

$$f''_{x,h}(t) - g''_n(t^+) = -\alpha(t)\varphi_{x,h}(t) + \langle R^{(0,2)}(\cdot, t^+), f_{x,h} - g_n \rangle. \tag{77}$$

Injecting (76) and (77) in (75) gives,

$$f_{x,h}(t) - g_n(t) = -\frac{1}{2}(t - t_{x,i}) d_{x,i}(f''_{x,h}(\sigma_{x,i}) - g''_n(\sigma^+_{x,i})) + \frac{1}{2}(t - t_{x_i})^2 (f''_{x,h}(\sigma_{x,i}) - g''_n(\sigma^+_{x,i}))$$
$$= \frac{1}{2} d_{x,i}(t - t_{x,i})\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i}) - \frac{1}{2}(t - t_{x,i})^2 \alpha(\sigma_{x,t})\varphi_{x,h}(\sigma_{x,t})$$
$$- \frac{1}{2} d_{x,i}(t - t_{x,i})\langle R^{(0,2)}(\cdot, \sigma^+_{x,i}), f_{x,h} - g_n \rangle + \frac{1}{2}(t - t_{x,i})^2 \langle R^{(0,2)}(\cdot, \sigma^+_{x,t}), f_{x,h} - g_n \rangle.$$

Thus,

$$\int_{t_{x,i}}^{t_{x,i+1}} (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t)\, dt =$$

$$\frac{1}{2} d_{x,i}\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})\varphi_{x,h}(t)\, dt - \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 \alpha(\sigma_{x,t})\varphi_{x,h}(\sigma_{x,t})\varphi_{x,h}(t)\, dt$$
$$- \frac{1}{2} d_{x,i}\langle R^{(0,2)}(\cdot, \sigma^+_{x,i}), f_{x,h} - g_n\rangle \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})\varphi_{x,h}(t)\, dt$$
$$+ \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 \langle R^{(0,2)}(\cdot, \sigma^+_{x,t}), f_{x,h} - g_n\rangle \varphi_{x,h}(t)\, dt$$
$$= \frac{1}{4} d^3_{x,i}\alpha(\sigma_{x,i})\varphi^2_{x,h}(\sigma_{x,i}) - \frac{1}{6} d^3_{x,i}\alpha(\sigma_{x,i})\varphi^2_{x,h}(\sigma_{x,i})$$
$$+ \frac{1}{2} d_{x,i}\alpha(\sigma_{x_i})\varphi_{x,h}(\sigma_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})[\varphi_{x,h}(t) - \varphi_{x,h}(\sigma_{x,i})]\, dt$$
$$- \frac{1}{2}\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 [\varphi_{x,h}(t) - \varphi_{x,h}(\sigma_{x,i})]\, dt$$
$$- \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 [\alpha(\sigma_{x,t})\varphi_{x,h}(\sigma_{x,t}) - \alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i})]\varphi_{x,h}(t)\, dt$$
$$- \frac{1}{2} d_{x,i}\langle R^{(0,2)}(\cdot, \sigma^+_{x,i}), f_{x,h} - g_n\rangle \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})\varphi_{x,h}(t)\, dt$$
$$+ \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 \langle R^{(0,2)}(\cdot, \sigma^+_{x,t}), f_{x,h} - g_n\rangle \varphi_{x,h}(t)\, dt$$
$$= \frac{1}{12} d^3_{x,i}\alpha(\sigma_{x,i})\varphi^2_{x,h}(\sigma_{x,i}) + A^{(1)}_{x,i} - A^{(2)}_{x,i} - A^{(3)}_{x,i} - A^{(4)}_{x,i} + A^{(5)}_{x,i}, \tag{78}$$

where,
$$A^{(1)}_{x,i} = \frac{1}{2} d_{x,i}\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})[\varphi_{x,h}(t) - \varphi_{x,h}(\sigma_{x,i})]\, dt.$$
$$A^{(2)}_{x,i} = \frac{1}{2}\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 [\varphi_{x,h}(t) - \varphi_{x,h}(\sigma_{x,i})]\, dt.$$
$$A^{(3)}_{x,i} = \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 [\alpha(\sigma_{x,t})\varphi_{x,h}(\sigma_{x,t}) - \alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i})]\varphi_{x,h}(t)\, dt.$$
$$A^{(4)}_{x,i} = \frac{1}{2} d_{x,i}\langle R^{(0,2)}(\cdot, \sigma^+_{x,i}), f_{x,h} - g_n\rangle \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})\varphi_{x,h}(t)\, dt.$$
$$A^{(5)}_{x,i} = \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 \langle R^{(0,2)}(\cdot, \sigma^+_{x,t}), f_{x,h} - g_n\rangle \varphi_{x,h}(t)\, dt.$$

We shall now control these quantities. Let,

$$B^{(1)}_{x,i} = \sup_{t_{x,i} < s,t < t_{x,i+1}} |\varphi_{x,h}(t) - \varphi_{x,h}(s)| \text{ and } B^{(2)}_{x,i} = \sup_{t_{x,i} < s,t < t_{x,i+1}} |\alpha(t)\varphi_{x,h}(t) - \alpha(s)\varphi_{x,h}(s)|.$$

Since α and $\varphi_{x,h}$ are Lipschitz then,

$$\sup_{0\le i\le n} B^{(1)}_{x,i} = O\left(\frac{1}{h^2}\sup_{0\le j\le n} d_{j,n}\right) \quad \text{and} \quad \sup_{0\le i\le n} B^{(2)}_{x,i} = O\left(\frac{1}{h^2}\sup_{0\le j\le n} d_{j,n}\right). \tag{79}$$

Elementary calculations show that,

$$|A^{(1)}_{x,i}| \leq \frac{a_1}{h} B^{(1)}_{x,i} d^3_{x,i}, \qquad |A^{(2)}_{x,i}| \leq \frac{a_2}{h} B^{(1)}_{x,i} d^3_{x,i} \quad \text{and} \quad |A^{(3)}_{x,i}| \leq \frac{a_3}{h} B^{(2)}_{x,i} d^3_{x,i}, \tag{80}$$

for appropriate constants a_1, a_2 and a_3. We obtain from the Cauchy-Schwartz inequality, Assumption (C) and Proposition 5 that,

$$|A^{(4)}_{x,i}| + |A^{(5)}_{x,i}| \leq \frac{a_4}{h} d^3_{x,i} ||f_{x,h} - g_n|| \leq \frac{1}{h} d^3_{x,i} \underbrace{a_4 \sqrt{\frac{C}{h}}}_{a_h} \sup_{0\leq j\leq n} d_{j,n}, \tag{81}$$

for an appropriate constant a_4 (C is defined in Proposition 5). Thus,

$$\begin{aligned}
&\int_{t_{x,i}}^{t_{x,i+1}} (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t)\, dt \\
&= \frac{1}{12} d^3_{x,i} \alpha(\sigma_{x,i}) \varphi^2_{x,h}(\sigma_{x,i}) + A^{(1)}_{x,i} - A^{(2)}_{x,i} - A^{(3)}_{x,i} - A^{(4)}_{x,i} + A^{(5)}_{x,i} \\
&\geq \frac{1}{12} d^3_{x,i} \alpha(\sigma_{x,i}) \varphi^2_{x,h}(\sigma_{x,i}) - d^3_{x,i}\left(\frac{a_1}{h} B^{(1)}_{x,i} + \frac{a_2}{h} B^{(2)}_{x,i} + \frac{a_h}{h} \sup_{0\leq j\leq n} d_{j,n}\right).
\end{aligned} \tag{82}$$

Let,

$$\rho_{h,N_{T_n}} = \sup_{0\leq i\leq N_{T_n}} \left(\frac{a_1}{h} B^{(1)}_{x,i} + \frac{a_2}{h} B^{(2)}_{x,i} + \frac{a_h}{h} \sup_{0\leq j\leq n} d_{j,n}\right).$$

Equation (79) implies that for an appropriate constant c and c' we have,

$$|\rho_{h,N_{T_n}}| \leq \left(\frac{c}{h^3} \sup_{0\leq j\leq n} d_{j,n} + \frac{c'}{h^{3/2}} \sup_{0\leq j\leq n} d_{j,n}\right).$$

Using (82) and (74) together with Equation (82) in (74) we obtain,

$$\begin{aligned}
||f_{x,h} - g_n||^2 &\geq \sum_{i=1}^{N_{T_n}-1} \left(\frac{1}{12} \alpha(\sigma_{x,i}) \varphi^2_{x,h}(\sigma_{x,i}) - \rho_{h,N_{T_n}}\right) d^3_{x,i} \\
&\geq \frac{1}{12} \sum_{i=1}^{N_{T_n}-1} \alpha(\sigma_{x,i}) \varphi^2_{x,h}(\sigma_{x,i}) d^3_{x,i} - \frac{c N_{T_n}}{h^3} \sup_{0\leq j\leq n} d^4_{j,n} - \frac{c' N_{T_n}}{h^{3/2}} \sup_{0\leq j\leq n} d^4_{j,n}.
\end{aligned} \tag{83}$$

Then the Hölder's inequality gives,

$$||f_{x,h} - g_n||^2 \geq \frac{1}{12(N_{T_n}-1)^2} \left\{ \sum_{j=1}^{N_{T_n}-1} [\alpha(\sigma_{x,i}) \varphi^2_{x,h}(\sigma_{x,i})]^{\frac{1}{3}} d_{x,i} \right\}^3 - \frac{c N_{T_n}}{h^3} \sup_{0\leq j\leq n} d^4_{j,n} - \frac{c' N_{T_n}}{h^{3/2}} \sup_{0\leq j\leq n} d^4_{j,n}.$$

We shall now control the first term of the right side of this inequality. We have,

$$\Big\{\sum_{j=1}^{N_{T_n}-1}\Big(\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i})\Big)^{\frac{1}{3}}d_{x,i}\Big\}^3$$

$$\begin{aligned}
&=\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt-\sum_{j=1}^{N_{T_n}-1}\int_{t_{x,i}}^{t_{x,i+1}}\Big((\alpha(t)\varphi_{x,h}^2(t))^{\frac{1}{3}}-(\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i}))^{\frac{1}{3}}\Big)\Big\}^3\\
&=\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt\Big\}^3-\Big\{\sum_{j=1}^{N_{T_n}-1}\int_{t_{x,i}}^{t_{x,i+1}}\Big((\alpha(t)\varphi_{x,h}^2(t))^{\frac{1}{3}}-(\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i}))^{\frac{1}{3}}\Big)\Big\}^3\\
&\quad-3\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt\Big\}^2\Big\{\sum_{j=1}^{N_{T_n}-1}\int_{t_{x,i}}^{t_{x,i+1}}\Big((\alpha(t)\varphi_{x,h}^2(t))^{\frac{1}{3}}-(\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i}))^{\frac{1}{3}}\Big)\Big\}\\
&\quad+3\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt\Big\}\Big\{\sum_{j=1}^{N_{T_n}-1}\int_{t_{x,i}}^{t_{x,i+1}}\Big((\alpha(t)\varphi_{x,h}^2(t))^{\frac{1}{3}}-(\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i}))^{\frac{1}{3}}\Big)\Big\}^2\\
&\triangleq\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt\Big\}^3+B,
\end{aligned}$$

We obtain using (47) and the fact that α is Lipschitz,

$$B=O\bigg(\Big(\frac{N_{T_n}}{h^{5/3}}\sup_{0\le j\le n}d_{j,n}^2\Big)^3\bigg)+O\bigg(\Big(\frac{N_{T_n}}{h^{5/3}}\sup_{0\le j\le n}d_{j,n}^2\Big)\,h^{\,2/3}\bigg)+O\bigg(\Big(\frac{N_{T_n}}{h^{5/3}}\sup_{0\le j\le n}d_{j,n}^2\Big)^{\,2}\,h^{1/3}\bigg).$$

Assumption (E) implies that for an appropriate constant c'' we have,

$$|B|\le\frac{c''N_{T_n}}{h}\sup_{0\le j\le n}d_{j,n}^2.$$

Using the Riemann integrability of α and $\varphi_{x,h}$ we get,

$$\begin{aligned}
&||f_{x,h}-g_n||^2\ge\frac{1}{12(N_{T_n}-1)^2}\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt\Big\}^3-\frac{c''}{N_{T_n}h}\sup_{0\le j\le n}d_{j,n}^2\\
&\quad-\frac{cN_{T_n}}{h^3}\sup_{0\le j\le n}d_{j,n}^{\,4}-\frac{c'N_{T_n}}{h^{3/2}}\sup_{0\le j\le n}d_{j,n}^4\\
&\ge\frac{1}{12N_{T_n}^2}\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt\Big\}^3-\frac{c''}{N_{T_n}h}\sup_{0\le j\le n}d_{j,n}^2-\frac{cN_{T_n}}{h^3}\sup_{0\le j\le n}d_{j,n}^{\,4}-\frac{c'N_{T_n}}{h^{3/2}}\sup_{0\le j\le n}d_{j,n}^4\\
&=\frac{1}{12h^2N_{T_n}^2}\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)K^2(\frac{x-t}{h})\Big)^{\frac{1}{3}}\,dt\Big\}^3-\frac{c''}{N_{T_n}h}\sup_{0\le j\le n}d_{j,n}^2\\
&\quad-\frac{cN_{T_n}}{h^3}\sup_{0\le j\le n}d_{j,n}^{\,4}-\frac{c'N_{T_n}}{h^{3/2}}\sup_{0\le j\le n}d_{j,n}^4\\
&=\frac{h}{12N_{T_n}^2}\Big\{\int_{-1}^{1}\Big(\alpha(x-th)K^2(t)\Big)^{\frac{1}{3}}\,dt\Big\}^3-\frac{c''}{N_{T_n}h}\sup_{0\le j\le n}d_{j,n}^2\\
&\quad-\frac{cN_{T_n}}{h^3}\sup_{0\le j\le n}d_{j,n}^{\,4}-\frac{c'N_{T_n}}{h^{3/2}}\sup_{0\le j\le n}d_{j,n}^4.
\end{aligned}$$

Assumption (E) implies that,

$$\lim_{n\to\infty}\frac{1}{h^2}N_{T_n}\sup_{0\le j\le n}d_{j,n}^2=0,\quad \lim_{n\to\infty}\frac{1}{h^4}\sup_{0\le j\le n}d_{j,n}^{\,4}N_{T_n}^3=0\quad\text{and}\quad\lim_{n\to\infty}\frac{c'}{h^{3/2}}\sup_{0\le j\le n}d_{j,n}^4N_{T_n}=0.$$

Finally the continuity of α yields,

$$\underline{\lim}_{n\to\infty} \frac{N_{T_n}^2}{h}||f_{x,h} - g_n||^2 \geq \frac{1}{12}\alpha(x)\left\{\int_{-1}^{1} K(t)^{\frac{2}{3}}\, dt\right\}^3.$$

Inequality (73) is then proved for a sequence of designs containing $x-h$ and $x+h$. Consider now any sequence of designs $\{T_n, n \geq 1\}$ satisfying Assumption (E) we can adjoin the points $\{x-h, x+h\}$ to T_n (if they aren't present). Hence we form a sequence $\{S_n, n \geq 1\}$ with $S_n \in D_{n+2}$ and satisfying (73). We have,

$$||f_{x,h} - P_{|S_n} f_{x,h}||^2 \leq ||f_{x,h} - P_{|T_n} f_{x,h}||^2.$$

Then,

$$N_{S_n}^2 ||f_{x,h} - P_{|S_n} f_{x,h}||^2 \leq N_{S_n}^2 ||f_{x,h} - P_{|T_n} f_{x,h}||^2. \tag{84}$$

We know that $N_{S_n} \in \{N_{T_n}+1, N_{T_n}+2\}$, replacing N_{S_n} in the right term of (84) by $(N_{T_n}+2)$ (or $(N_{T_n}+1)$) gives,

$$\frac{N_{S_n}^2}{h}||f_{x,h} - P_{|S_n} f_{x,h}||^2 - \frac{(4+2N_{T_n})}{h}||f_{x,h} - P_{|T_n} f_{x,h}||^2 \leq \frac{N_{T_n}^2}{h}||f_{x,h} - P_{|T_n} f_{x,h}||^2.$$

Assumption (E) and Equation (51) yield,

$$\lim_{n\to\infty} \frac{(4+2N_{T_n})}{h}||f_{x,h} - P_{|T_n} f_{x,h}||^2 = 0.$$

Hence, for any sequence $\{T_n, n \geq 1\}$ we have,

$$\underline{\lim}_{n\to\infty} \frac{N_{T_n}^2}{h}||f_{x,h} - P_{|T_n} f_{x,h}||^2 \geq \frac{1}{12}\alpha(x)\left\{\int_{-1}^{1} K^{2/3}(t)dt\right\}^3.$$

This completes the proof of Proposition 6. □

6.9 Proof of Proposition 7.

On the one hand, Proposition 5 yields that there exists a constant $c > 0$ such that,

$$0 \leq \frac{\sigma_{x,h}^2}{m} - \mathrm{Var}\ \hat{g}_n^{pro}(x) \leq \frac{c}{mh} \sup_{0\leq j\leq n} d_{j,n}^2.$$

Lemma 2 implies that there exists a constant $c' > 0$ such that,

$$\sup_{0\leq j\leq n} d_{j,n}^2 \leq \frac{c'}{n^2}.$$

Thus, for $n \geq 1$ we have,

$$0 \leq \frac{\sigma_{x,h}^2}{m} - \mathrm{Var}\ \hat{g}_n^{pro}(x) \leq \frac{c'c}{mn^2h}.$$

Finally, taking $C = cc'$ we obtain,

$$\overline{\lim}_{n\to\infty} mn^2h\Big(\frac{\sigma_{x,h}^2}{m} - \mathrm{Var}\ \hat{g}_n^{pro}(x)\Big) \leq C.$$

Inequality (10) is then proved. On the other hand, Proposition 6 yields,

$$\frac{mN_{T_n}^2}{h}\Big(\frac{\sigma_{x,h}^2}{m} - \text{Var } \hat{g}_n^{pro}(x)\Big) \geq \frac{1}{12}\alpha(x)\Big\{\int_{-1}^{1} K^{2/3}(t)dt\Big\}^3.$$

Lemma 2 implies that there exists a constant $c'' > 0$ such that,

$$N_{T_n} < c''nh,$$

which implies that,

$$c''mn^2h\,\Big(\frac{\sigma_{x,h}^2}{m} - \text{Var } \hat{g}_n^{pro}(x)\Big) \geq \frac{1}{12}\alpha(x)\Big\{\int_{-1}^{1} K^{2/3}(t)dt\Big\}^3.$$

Finally, taking $C' = \frac{1}{12c''}\alpha(x)\Big\{\int_{-1}^{1} K^{2/3}(t)dt\Big\}^3$ we obtain,

$$\underline{\lim}_{n\to\infty}\; mn^2h\Big(\frac{\sigma_{x,h}^2}{m} - \text{Var } \hat{g}_n^{pro}(x)\Big) \geq C'.$$

This concludes the proof of Proposition 7. □

6.10 Proof of Proposition 8

The first part of this proof is the same as that of Proposition (6). Recall that,

$$m\Big(\frac{\sigma_{x,h}^2}{m} - \text{Var } \hat{g}_n^{pro}(x)\Big) = ||f_{x,h}||^2 - ||P_{|T_n}f_{x,h}||^2 = ||f_{x,h} - P_{|T_n}f_{x,h}||^2.$$

Using (74) and (78) we obtain,

$$\begin{aligned}\text{Var } \hat{g}_n^{pro}(x) - \frac{\sigma_{x,h}^2}{m} &= -\frac{1}{m}||f_{x,h} - P_{|T_n}f_{x,h}||^2\\ &= -\frac{1}{m}\sum_{i=1}^{N_{T_n}}\Big(\frac{1}{12}d_{x,i}^3\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i}) + A_{x,i}^{(1)} - A_{x,i}^{(2)} - A_{x,i}^{(3)} - A_{x,i}^{(4)} + A_{x,i}^{(5)}\Big),\end{aligned} \tag{85}$$

for some $\sigma_{x,i} \in]t_{x,i}, t_{x,i+1}[$ and some $\sigma_{x,t} \in]t_{x,i}, t[$, where,

$$A_{x,i}^{(1)} = \frac{1}{2}d_{x,i}\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i})\int_{t_{x,i}}^{t_{x,i+1}}(t - t_{x,i})[\varphi_{x,h}(t) - \varphi_{x,h}(\sigma_{x,i})]\;dt.$$

$$A_{x,i}^{(2)} = \frac{1}{2}\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i})\int_{t_{x,i}}^{t_{x,i+1}}(t - t_{x,i})^2[\varphi_{x,h}(t) - \varphi_{x,h}(\sigma_{x,i})]\;dt.$$

$$A_{x,i}^{(3)} = \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}}(t - t_{x,i})^2[\alpha(\sigma_{x,t})\varphi_{x,h}(\sigma_{x,t}) - \alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i})]\varphi_{x,h}(t)\;dt.$$

$$A_{x,i}^{(4)} = \frac{1}{2}d_{x,i}\langle R^{(0,2)}(\cdot,\sigma_{x,i}^+), f_{x,h} - g_n\rangle\int_{t_{x,i}}^{t_{x,i+1}}(t - t_{x,i})\varphi_{x,h}(t)\;dt.$$

$$A_{x,i}^{(5)} = \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}}(t - t_{x,i})^2\langle R^{(0,2)}(\cdot,\sigma_{x,t}^+), f_{x,h} - g_n\rangle\varphi_{x,h}(t)\;dt.$$

From the definition of the regular sequence of designs (see Definition 2) and the mean value theorem we have for $i = 1, \cdots, N_{T_n}$,

$$d_{x,i} = t_{x,i+1} - t_{x,i} = F^{-1}\Big(\frac{i+1}{n}\Big) - F^{-1}\Big(\frac{i}{n}\Big) = \frac{1}{nf(t^*_{x,i})},$$

where $t^*_{x,i} \in]t_{x,i}, t_{x,i+1}[$. Using this together with (85) we obtain,

$$\begin{aligned}\text{Var } \hat{g}_n^{pro}(x) - \frac{\sigma^2_{x,h}}{m} &= -\frac{1}{12mn^2}\sum_{i=1}^{N_{T_n}} d_{x,i}\frac{1}{f^2(t^*_{x,i})}\alpha(\sigma_{x,i})\varphi^2_{x,h}(\sigma_{x,i}) \\ &\quad - \frac{1}{m}\sum_{i=1}^{N_{T_n}}\Big(A^{(1)}_{x,i} - A^{(2)}_{x,i} - A^{(3)}_{x,i} - A^{(4)}_{x,i} + A^{(5)}_{x,i}\Big).\end{aligned}$$

Lemma 2 yields that $N_{T_n} = O(nh)$. Using (80), (81) and (79) we obtain,

$$A^{(1)}_{x,i} = O\Big(\frac{1}{n^4h^3}\Big),\ \ A^{(2)}_{x,i} = O\Big(\frac{1}{n^4h^3}\Big),\ \ A^{(3)}_{x,i} = O\Big(\frac{1}{n^4h^3}\Big) \text{ and } A^{(4)}_{x,i} + A^{(5)}_{x,i} = O\Big(\frac{1}{n^4h^{3/2}}\Big).$$

Finally,

$$\text{Var } \hat{g}_n^{pro}(x) - \frac{\sigma^2_{x,h}}{m} = -\frac{1}{12mn^2}\sum_{i=1}^{N_{T_n}} d_{x,i}\frac{1}{f^2(t^*_{x,i})}\alpha(\sigma_{x,i})\varphi^2_{x,h}(\sigma_{x,i}) + O\Big(\frac{1}{mn^3h^2} + \frac{1}{mn^3\sqrt{h}}\Big).$$

Using a classical approximation of a sum by an integral (see for instance, Lemma 2 in [7]) and the fact that $0 < h < 1$ we obtain,

$$\text{Var } \hat{g}_n^{pro}(x) - \frac{\sigma^2_{x,h}}{m} = -\frac{1}{12mn^2}\int_{x-h}^{x+h}\frac{\alpha(t)}{f^2(t)}\varphi^2_{x,h}(t)\,dt + O\Big(\frac{1}{mn^3h^2}\Big).$$

This concludes the proof of Proposition 8. □

6.11 Proof of Theorem 2.

First, note that since α and f are Lipschitz functions then the asymptotic expression of the integral in (12) is:

$$\begin{aligned}\frac{1}{mn^2}\int_{x-h}^{x+h}\frac{\alpha(t)}{f^2(t)}\varphi^2_{x,h}(t)dt &= \frac{1}{mn^2h}\int_{-1}^{1}\frac{\alpha(x-th)}{f^2(x-th)}K^2(t)\,dt \\ &= \frac{1}{mn^2h}\Big(\frac{\alpha(x)}{f^2(x)}\int_{-1}^{1}K^2(t)\,dt + \int_{-1}^{1}\Big(\frac{\alpha(x-th)}{f^2(x-th)} - \frac{\alpha(x)}{f^2(x)}\Big)K^2(t)\,dt\Big) \\ &= \frac{1}{mn^2h}\frac{\alpha(x)}{f^2(x)}\int_{-1}^{1}K^2(t)\,dt + O\Big(\frac{1}{mn^2}\Big).\end{aligned}$$

This last equality together with Proposition 8 and Proposition 4 concludes the proof of Theorem 2. □

6.12 Proof of Corollary 1.

Let $I_1 = \int_0^1 R(x,x)w(x)\,dx$ and put,

$$\Psi(h,m) = -\frac{C_K h}{2m}\int_0^1 \alpha(x)w(x)\,dx + \frac{1}{4}h^4 B^2 \int_0^1 [g''(x)]^2 w(x)\,dx.$$

We have from Theorem 1,

$$\mathrm{IMSE}(h) = \frac{I}{m} + \Psi(h,m) + o\Big(h^4 + \frac{h}{m}\Big) + O\Big(\frac{1}{mn^2h} + \frac{h}{n} + \frac{1}{n^2h^2}\Big),$$

Let h^* be as defined by (14). It is clear that $h^* = \underset{0<h<1}{\operatorname{argmin}}\ \Psi(h,m)$ so that $\Psi(h,m) \geq \Psi(h^*,m)$ for every $0 < h < 1$. Let $h_{n,m}$ be as defined in Corollary 1. We have,

$$\frac{\mathrm{IMSE}(h^*)}{\mathrm{IMSE}(h_{n,m})} = \frac{\frac{I_1}{m} + \Psi(h^*,m) + o\Big(h^{*4} + \frac{h^*}{m}\Big) + O\Big(\frac{1}{mn^2h^*} + \frac{h^*}{n} + \frac{1}{n^2h^{*2}}\Big)}{\frac{I_1}{m} + \Psi(h_{n,m},m) + o\Big(h_{n,m}^4 + \frac{h_{n,m}}{m}\Big) + O\Big(\frac{1}{mn^2h_{n,m}} + \frac{h_{n,m}}{n} + \frac{1}{n^2h_{n,m}^2}\Big)}$$

$$\leq \frac{I_1 + m\Psi(h_{n,m},m) + o\Big(mh^{*4} + h^*\Big) + O\Big(\frac{1}{n^2h^*} + \frac{mh^*}{n} + \frac{m}{n^2h^2}\Big)}{I_1 + m\Psi(h_{n,m},m) + o\Big(mh_{n,m}^4 + h_{n,m}\Big) + O\Big(\frac{1}{n^2h_{n,m}} + \frac{mh_{n,m}}{n} + \frac{m}{n^2h_{n,m}^2}\Big)}.$$

We have, using the definition of h^*, $mh_{n,m}^3 = O(1)$, $\lim_{n,m\to\infty} h_{n,m} = 0$ and using the assumption $\frac{m}{n} = O(1)$ as $n,m \to \infty$ we know that $m\Psi(h_{n,m},m) = O(h_{n,m})$. Thus,

$$\overline{\lim_{n,m\to\infty}} \frac{\mathrm{IMSE}(h^*)}{\mathrm{IMSE}(h_{n,m})} \leq 1.$$

This concludes the proof of Corollary 1. □

6.13 Proof of Theorem 3.

Let $x \in]0,1[$ be fixed. We have the following decomposition,

$$\sqrt{m}\Big(\hat{g}_{n,m}^{pro}(x) - g(x)\Big) = \sqrt{m}\Big(\hat{g}_{n,m}^{pro}(x) - \mathbb{E}\big(\hat{g}_{n,m}^{pro}(x)\big)\Big) + \sqrt{m}\ \Big(\mathbb{E}(\hat{g}_{n,m}^{pro}(x)) - g(x)\Big). \tag{86}$$

Since $\lim_{n,m\to\infty} \sqrt{m}h = 0$, $\frac{n}{m} = O(1)$ as $n,m \to \infty$ and $\lim_{n,m\to\infty} nh^2 = \infty$ then Remark 4 implies that,

$$\lim_{n,m\to\infty} \sqrt{m}\ \Big(\mathbb{E}(\hat{g}_{n,m}^{pro}(x)) - g(x)\Big) = 0. \tag{87}$$

Consider now the first term of the right side of (86). Since $\overline{Y}(t_{x,i}) - \mathbb{E}(\overline{Y}(t_{x,i})) = \overline{\varepsilon}(t_{x,i})$, we have, as done by [14],

$$\sqrt{m}\Big(\hat{g}_{n,m}^{pro}(x) - \mathbb{E}\big(\hat{g}_{n,m}^{pro}(x)\big)\Big) = \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\sum_{i=1}^{n} m_{x,h}(t_i)\varepsilon_j(t_i)$$

$$= \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\sum_{i=1}^{n} m_{x,h}(t_i)\big(\varepsilon_j(t_i) - \varepsilon_j(x)\big) + \Big(\sum_{i=1}^{n} m_{x,h}(t_i)\Big)\Big(\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\varepsilon_j(x)\Big). \tag{88}$$

We start by controlling the second term of this last equation. Using Lemma 3 together with Lemma 2 we obtain,

$$m_{x,h}(t_{i,n}) = \begin{cases} \frac{1}{2}\varphi_{x,h}(t_{i,n})(t_{i+1,n}-t_{i-1,n}) + O\Big(\frac{1}{n^2h^2}+\frac{1}{n^2\sqrt{h}}\Big) & if \quad i \notin \{1,n\} \text{ and} \\ & [t_{i-1,n},t_{i+1,n}]\cap[x-h,x+h]\neq\emptyset, \\ O\Big(\frac{1}{n^2h^2}+\frac{1}{n^2\sqrt{h}}\Big) & \text{if} \quad i\in\{1,n\}, \\ O\Big(\frac{1}{n^2\sqrt{h}}\Big) & otherwise. \end{cases}$$

Recall that $N_{T_n} = \text{Card } I_{x,h} = \text{Card }\{i=1,\cdots,n \;:\; [t_{i-1},t_{i+1}]\cap]x-h,x+h[\neq\emptyset\}$ and denote by $t_{x,i}$ the points of T_n for which $i \in I_{x,h}$, Lemma 2 yields that $N_{T_n} = O(nh)$. Thus,

$$\sum_{i=1}^{n} m_{x,h}(t_i) = \frac{1}{2}\sum_{i=2}^{N_{T_n}-1}\varphi_{x,h}(t_{x,i})(t_{x,i+1}-t_{x,i-1}) + O\Big(\frac{1}{nh}\Big).$$

Since $\lim\limits_{n\to\infty} nh = +\infty$, then using the Riemann integrability of K, we obtain,

$$\lim_{n,m\to\infty}\sum_{i=1}^{n} m_{x,h}(t_i) = \frac{1}{2}\lim_{n,m\to\infty}\sum_{i=2}^{N_{T_n}-1}\varphi_{x,h}(t_{x,i})(t_{x,i+1}-t_{x,i-1}) = \int_{-1}^{1}K(t)\;dt = 1.$$

The Central Limit Theorem for i.i.d. variables yields,

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\varepsilon_j(x) \xrightarrow[m\to\infty]{\mathscr{D}} Z \quad where \quad Z\sim\mathcal{N}(0,R(x,x)).$$

We shall prove now that the first term of Equation (88) tends to 0 in probability as n,m tends to infinity. Let,

$$A_{m,n}(x) = \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\sum_{i=1}^{n} m_{x,h}(t_i)\big(\varepsilon_j(t_i)-\varepsilon_j(x)\big) \triangleq \frac{1}{\sqrt{m}}\sum_{j=1}^{m}T_{n,j}(x).$$

From the Chebyshev inequality, it suffices to prove that $\lim\limits_{n,m\to\infty}\mathbb{E}(A^2_{m,n}(x)) = 0$. We have for $j\neq l$, $\mathbb{E}(\varepsilon_j(x)\varepsilon_l(y)) = 0$ then $\mathbb{E}(T_{n,j}(x)T_{n,l}(x)) = 0$. Hence,

$$\mathbb{E}(A^2_{m,n}(x)) = \frac{1}{m}\sum_{j=1}^{m}\sum_{l=1}^{m}\mathbb{E}(T_{n,j}(x)T_{n,l}(x)) = \frac{1}{m}\sum_{j=1}^{m}\mathbb{E}(T^2_{n,j}(x)).$$

We have,

$$\begin{aligned}\mathbb{E}(T^2_{n,j}(x)) &= \sum_{i=1}^{n}\sum_{k=1}^{n} m_{x,h}(t_i)m_{x,h}(t_k)\mathbb{E}\Big(\big(\varepsilon_j(t_i)-\varepsilon_j(x)\big)\big(\varepsilon_j(t_k)-\varepsilon_j(x)\big)\Big) \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} m_{x,h}(t_i)m_{x,h}(t_k)\Big(R(t_i,t_k)-R(t_i,x)-R(x,t_k)+R(x,x)\Big).\end{aligned}$$

Note that $\mathbb{E}(T^2_{n,j}(x))$ does not depend on j hence,

$$\begin{aligned}\mathbb{E}(A^2_{m,n}(x)) &= \sum_{i=1}^{n}\sum_{k=1}^{n} m_{x,h}(t_i)m_{x,h}(t_k)\Big(R(t_i,t_k) - R(t_i,x) - R(x,t_k) + R(x,x)\Big)\\ &\triangleq B_{n,1}(x) - B_{n,2}(x) - B_{n,3}(x) + B_{n,4}(x). \qquad (89)\end{aligned}$$

Using Lemma 3 and the approximation of a sum by an integral (see, for instance, Lemma 2 in [7]) we obtain,

$$B_{n,1}(x) = \int_{x-h}^{x+h}\int_{x-h}^{x+h} \varphi_{x,h}(s)\varphi_{x,h}(t)R(s,t)\ ds\ dt + O\Big(\frac{1}{nh}\Big) = \sigma^2_{x,h} + O\Big(\frac{1}{nh}\Big).$$

Using Equation (13) we obtain,

$$B_{n,1}(x) = R(x,x) - \frac{1}{2}\alpha(x)C_K h + o(h) + O\Big(\frac{1}{nh}\Big).$$

where $C_K = \int_{-1}^{1}\int_{-1}^{1}|u-v|K(u)K(v)dudv$. Since $\lim_{n\to\infty} h = 0$ and $\lim_{n\to\infty} nh = \infty$ then,

$$\lim_{n\to\infty} B_{n,1}(x) = R(x,x). \qquad (90)$$

Consider now the term $B_{n,2}(x)$. We obtain using Lemma 3 and the approximation of a sum by an integral,

$$\begin{aligned}B_{n,2}(x) &= \int_{x-h}^{x+h}\int_{x-h}^{x+h} \varphi_{x,h}(s)\varphi_{x,h}(t)R(s,x)\ ds\ dt + O\Big(\frac{1}{nh}\Big)\\ &= \int_{x-h}^{x+h} \varphi_{x,h}(s)R(s,x)\ ds + O\Big(\frac{1}{nh}\Big)\\ &= \int_{-1}^{1} K(s)R(x-hs,x)\ ds + O\Big(\frac{1}{nh}\Big)\\ &= \int_{-1}^{0} K(s)R(x-hs,x)\ ds + \int_{0}^{1} K(s)R(x-hs,x)\ ds + O\Big(\frac{1}{nh}\Big).\end{aligned}$$

For $s \in]-1,0[$, Taylor expansion of $R(\cdot,x)$ around x yields,

$$R(s,x) = R(x-sh,x) - shR^{(1,0)}(x^+,x) + o(h).$$

Similarly for $s \in]0,1[$ we obtain,

$$R(x-sh,x) = R(x,x) - shR^{(1,0)}(x^-,x) + o(h).$$

Thus,

$$B_{n,2}(x) = R(x,x) - hR^{(1,0)}(x^+,x)\int_{-1}^{0} s\ k(s)\ ds - hR^{(1,0)}(x^-,x)\int_{0}^{1} s\ k(s)\ ds + o(h) + O\Big(\frac{1}{nh}\Big).$$

Hence,

$$\lim_{n\to\infty} B_{n,2}(x) = R(x,x). \qquad (91)$$

Similarly,

$$\lim_{n\to\infty} B_{n,3}(x) = R(x,x). \qquad (92)$$

It is easy to verify that,

$$\lim_{n\to\infty} B_{n,4}(x) = R(x,x). \tag{93}$$

Inserting (90), (91), (92) and (93) in (89) yields,

$$\lim_{n,m\to\infty} \mathbb{E}(A^2_{m,n}(x)) = 0.$$

This concludes the proof of Theorem 3. □

6.14 Proof of Theorem 5.

Let $x \in]0,1[$. On the one hand, we have from Proposition 8 and Remark 5,

$$\text{Var } \hat{g}_n^{pro}(x) = \frac{\sigma^2_{x,h}}{m} - \frac{A}{12mn^2h}\frac{\alpha(x)}{f^2(x)} + O\Big(\frac{1}{mn^3h^2} + \frac{1}{mn^2}\Big), \tag{94}$$

where $A = \int_{-1}^{1} K^2(t)\, dt$. On the other hand, it can be seen in [8] that,

$$\text{Var } \hat{g}_n^{GM}(x) = \frac{\sigma^2_{x,h}}{m} + O\Big(\frac{1}{mn^2} + \frac{1}{mn^3h^2}\Big). \tag{95}$$

Equations (94) and (95) then yield,

$$mn^2h\,\Big(\text{Var } \hat{g}_n^{GM} - \text{Var } \hat{g}_n^{pro}\Big) = \frac{A}{12}\frac{\alpha(x)}{f^2(x)} + O\Big(h + \frac{1}{nh}\Big).$$

Recall that $\alpha(x) > 0$ and that $\frac{1}{f(x)} > 0$. Since $h \to 0$ and $nh \to \infty$ as $n, m \to \infty$ we obtain,

$$\lim_{n,m\to\infty} mn^2h\,\Big(\text{Var } \hat{g}_n^{GM}(x) - \text{Var } \hat{g}_n^{pro}(x)\Big) = \frac{A}{12}\frac{\alpha(x)}{f^2(x)} > 0.$$

This concludes the proof of Theorem 5. □

6.15 Proof of Theorem 6.

We have from the proof of Proposition 4 (Equation (49)) for any $x \in]0,1[$,

$$\mathbb{E}(\hat{g}_{n,m}^{pro}(x)) - g(x) = I_h(x) - g(x) + O\Big(\frac{1}{n^2h}\Big), \tag{96}$$

where,

$$I_h(x) = \int_{x-h}^{x+h} \varphi_{x,h}(s) g(s)\, ds.$$

Hence, using (94) and (96) we get for a positive density measure w,

$$\begin{aligned}\text{IMSE}(\hat{g}_n^{pro}) = &\frac{1}{m}\int_0^1 \sigma^2_{x,h}\, w(x)\, dx - \frac{A}{12mn^2h}\int_0^1 \frac{\alpha(x)}{f^2(x)}\, w(x)\, dx + \int_0^1 \big(I_h(x) - g(x)\big)^2\, w(x)\, dx \\ &+ O\Big(\frac{1}{n^4h^2} + \frac{h}{n^2} + \frac{1}{mn^3h^2} + \frac{1}{mn^2}\Big).\end{aligned} \tag{97}$$

It can be seen in [8] that,

$$\mathbb{E}(\hat{g}_{n,m}^{GM}(x)) - g(x) = I_h(x) - g(x) + O\Big(\frac{1}{n^2h}\Big). \tag{98}$$

Using (95) and (98) yield,

$$\text{IMSE}(\hat{g}_n^{GM}) = \frac{1}{m}\int_0^1 \sigma_{x,h}^2\ w(x)\ dx + \int_0^1 \big(I_h(x) - g(x)\big)^2\ w(x)\ dx$$
$$+ O\Big(\frac{1}{n^4h^2} + \frac{h}{n^2} + \frac{1}{mn^2} + \frac{1}{mn^3h^2}\Big). \tag{99}$$

Then, Equations (97) and (99) yield,

$$mn^2h\left(\text{IMSE }(\hat{g}_n^{GM}) - \text{IMSE }(\hat{g}_n^{pro})\right) = \frac{A}{12}\int_0^1 \frac{\alpha(x)}{f^2(x)}\ w(x)\ dx + O\Big(\frac{m}{n^2h} + mh^2 + h + \frac{1}{nh}\Big).$$

Since $\frac{m}{n} = O(1)$ and $mh^2 \to 0$ as $n, m \to \infty$ we obtain,

$$\lim_{n,m\to\infty} mn^2h\left(\text{IMSE }(\hat{g}_n^{GM}) - \text{IMSE }(\hat{g}_n^{pro})\right) = \frac{A}{12}\int_0^1 \frac{\alpha(x)}{f^2(x)}\ w(x)\ dx > 0.$$

This concludes the proof of Theorem 6. □

7 Appendix

Let $\varepsilon = (\varepsilon(t))_{t\in[0,1]}$ be a centered and a second order process of autocovariance R, such that R is invertible when restricted to any finite set on $[0,1]$. Let $L(\varepsilon(t), t \in [0,1])$ be the set of all random variables which maybe be written as a linear combinations of $\varepsilon(t)$ for $t \in [0,1]$, i.e., the set of random variables of the form $\sum_{i=1}^{l} \alpha_i \varepsilon(t_i)$ for some positive integer l and some constants α_i, $t_i \in [0,1]$ for $i = 1, \cdots, l$. Let also $L_2(\varepsilon)$ be the Hilbert space of all square integrable random variables in the linear manifold $L(\varepsilon(t), t \in [0,1])$, together with all random variables U that are limits in $\mathbb{L}^2$ of a sequence of random variables U_n in $L(\varepsilon(t), t \in [0,1])$, i.e, U is such that,

$$\exists\, (U_n)_{n\geq 0} \in L(\varepsilon(t), t \in [0,1]) : \ \lim_{n\to\infty} \mathbb{E}((U_n - U)^2) = 0.$$

Denote by $\mathcal{F}(\varepsilon)$ the family of functions g on $[0,1]$ defined by,

$$\mathcal{F}(\varepsilon) = \{g : [0,1] \to \mathbb{R} \text{ with } g(\cdot) = \mathbb{E}(U\varepsilon(\cdot)) \ \text{ where } \ U \in L_2(\varepsilon)\},$$

We note here that for every $g \in \mathcal{F}(\varepsilon)$, the associated U is unique. It is easy to verify that $\mathcal{F}(\varepsilon)$ is a Hilbert space equipped with the norm $||\ ||$ defined for $g \in \mathcal{F}(\varepsilon)$ by,

$$||g||^2 = \mathbb{E}(U^2).$$

In fact, let $g \in \mathcal{F}(\varepsilon)$, i.e, $g(\cdot) = \mathbb{E}(U\varepsilon(\cdot))$ for some $U \in L_2(\epsilon)$. We have,

- $||g|| = \sqrt{\mathbb{E}(U^2)} \geq 0$.
- $||g|| = \sqrt{\mathbb{E}(U^2)} = 0 \Rightarrow U = 0$ a.s. $\Rightarrow g = 0$.
- For $g \in \mathcal{F}(\varepsilon)$, i.e, $f(\cdot) = \mathbb{E}(V\varepsilon(\cdot))$ some $V \in L_2(\epsilon)$. We have,

$$||g+f||^2 = \mathbb{E}((U+V)^2) = \mathbb{E}(U^2) + \mathbb{E}(V^2) + 2\mathbb{E}(UV)$$
$$\leq \mathbb{E}(U^2) + \mathbb{E}(V^2) + 2\sqrt{\mathbb{E}(U^2)}\sqrt{\mathbb{E}(V^2)} = \left(\sqrt{\mathbb{E}(U^2)} + \sqrt{\mathbb{E}(V^2)}\right)^2.$$

Thus, $||g+f|| \leq \sqrt{\mathbb{E}(U^2)} + \sqrt{\mathbb{E}(V^2)} = ||g|| + ||f||$.

We now prove the completeness of $\mathcal{F}(\varepsilon)$. For this let $g_n(\cdot) = E(U_n\varepsilon(\cdot))$ be a Cauchy sequence in $\mathcal{F}(\varepsilon)$, i.e.,

$$\lim_{n,m\to\infty} ||g_n - g_m||^2 = 0.$$

From the definition of the norm $|| \;||$ we obtain,

$$\lim_{n,m\to\infty} \mathbb{E}((U_n - U_m)^2) = \lim_{n,m\to\infty} ||g_n - g_m||^2 = 0.$$

This yields that $(U_n)_{n\geq 1}$ is a Cauchy sequence in $L_2(\varepsilon)$, which is a Hilbert space as proven by [22] (see page 8 there). Thus it exists $U \in L_2(\varepsilon)$ such that,

$$\lim_{n\to\infty} \mathbb{E}((U_n - U)^2) = 0.$$

Taking $g(\cdot) = \mathbb{E}(U\varepsilon(\cdot))$, which is clearly an element of $\mathcal{F}(\varepsilon)$ gives,

$$\lim_{n\to\infty} ||g_n - g||^2 = \lim_{n\to\infty} \mathbb{E}((U_n - U)^2) = 0.$$

This concludes the proof of completness of $\mathcal{F}(\varepsilon)$.

The Hilbert space $\mathcal{F}(\varepsilon)$ can easily be identified as the Reproducing Kernel Hilbert Space associated to a reproducing kernel R (with $R(s,t) = \mathbb{E}(\varepsilon(s)\varepsilon(t))$), which is defined as follows.

Definition 4 *[22] A Hilbert space H is said to be a Reproducing Kernel Hilbert Space associated to a reproducing kernel (or function) R (RKHS(R)), if its members are functions on some set T, and if there is a kernel R on $T \times T$ having the following two properties:*

$$\begin{cases} R(\cdot,t) \in H & \text{for all } \; t \in T, \\ \langle g, R(\cdot,t)\rangle = g(t) & \text{for all } \; t \in T \quad \text{and} \quad g \in H, \end{cases} \tag{100}$$

where $\langle \cdot,\cdot \rangle$ is the inner (or scalar) product in H.

To prove this, we need to verify the properties given in (100). For $t \in [0,1]$ we have,

$$R(s,t) = \mathbb{E}(\varepsilon(s)\varepsilon(t)) \qquad \text{for all } \; s \in [0,1].$$

Since $\varepsilon(s) \in L_2(\varepsilon)$ then $R(\cdot,t) \in \mathcal{F}(\varepsilon)$ for any fixed $t \in [0,1]$. Now let $g \in \mathcal{F}(\varepsilon)$, i.e.,

$$g(\cdot) = \mathbb{E}(U\varepsilon(\cdot)) \;\; \text{for some} \;\; U \in L_2(\varepsilon).$$

Then,

$$\begin{aligned} \langle g, R(\cdot,t)\rangle &= \frac{1}{2}\big(||g||^2 + ||R(\cdot,t)||^2 - ||g - R(\cdot,t)||^2\big) = \frac{1}{2}\big(\mathbb{E}(U^2) + \mathbb{E}(\varepsilon(t)^2) - \mathbb{E}((U - \varepsilon(t))^2)\big) \\ &= \frac{1}{2}\mathbb{E}(2U\varepsilon(t)) = g(t). \end{aligned}$$

These properties together with the following theorem yield that $\mathcal{F}(\varepsilon)$ is the RKHS(R).

Theorem 7 (E. H. Moor) *[3] A symmetric non-negative Kernel R generates a unique Hilbert space.*

In the sequel, we take R to be continuous on $[0,1]^2$ and we shall consider the function of interest given by (2). More generally, we consider the function f, defined for a continuous function φ and $t \in [0,1]$, by

$$f(t) = \int_0^1 R(s,t)\varphi(s)\ ds. \tag{101}$$

Lemma 5 *We have $f \in \mathcal{F}(\varepsilon)$, i.e., there exists $X \in L_2(\varepsilon)$ with,*

$$f(\cdot) = \mathbb{E}(X\varepsilon(\cdot)). \tag{102}$$

In addition,

$$\|f\|^2 = \mathbb{E}(X^2) = \int_0^1 \int_0^1 R(s,t)\varphi(s)\varphi(t)\ dt\ ds.$$

Proof. Define, for a suitable partition $(x_{i,n})_{i=1,\cdots,n}$ of $[0,1]$,

$$X_n = \sum_{i=1}^{n-1}(x_{i+1,n} - x_{i,n})\varphi(x_{i,n})\varepsilon(x_{i,n}) \in L_2(\varepsilon),$$

such that for any $t \in [0,1]$,

$$f(t) = \lim_{n\to\infty} \sum_{i=1}^{n-1}(x_{i+1,n} - x_{i,n})\varphi(x_{i,n})R(x_{i,n},t) = \lim_{n\to\infty} \mathbb{E}(X_n\varepsilon(t)).$$

We shall prove that $(X_n)_n$ converges to a certain element of $\mathbb{L}^2$, i.e.,

$$\exists\ X \in \mathbb{L}^2\ :\ \lim_{n\to\infty} \mathbb{E}\big((X_n - X)^2\big) = 0, \tag{103}$$

and by the definition of $L_2(\varepsilon)$ the limit in (103) proves that X is an element of $L_2(\varepsilon)$. Now the proof (103) is immediate, in fact it is easy to check that (X_n) id a Cauchy sequence in $\mathbb{L}^2$. By the completeness of $\mathbb{L}^2$, we deduce (103). In addition we have, $\lim_{n\to\infty} \mathbb{E}(X_n\varepsilon(t)) = \mathbb{E}(X\varepsilon(t))$, this is due to the following inequality,

$$\Big|\mathbb{E}(X_n\varepsilon(t)) - \mathbb{E}(X\varepsilon(t))\Big| \leq \mathbb{E}\Big|(X_n - X)\varepsilon(t)\Big| \leq \sqrt{\mathbb{E}((X_n - X)^2)}\sqrt{\mathbb{E}(\varepsilon(t)^2)},$$

and the fact that $\lim_{n\to\infty} \mathbb{E}((X_n - X)^2) = 0$ and $\mathbb{E}(\varepsilon(t)^2) < \infty$. The proof of (102) is concluded. Finally,

$$\begin{aligned}\mathbb{E}(X^2) = \lim_{n\to\infty} \mathbb{E}(X_n^2) &= \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n}(x_{i+1,n} - x_{i,n})(x_{j+1,n} - x_{j,n})\varphi(x_{i,n})\varphi(x_{j,n})R(x_{i,n}, x_{j,n}) \\ &= \int_0^1 \int_0^1 \varphi(t)\varphi(t)R(s,t)\ ds\ dt.\end{aligned}$$

This concludes the proof of Lemma 5. $\square$

Now let $T_n = (t_1, t_2, \cdots, t_n)$ with $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$ and let V_{T_n} be the subspace of $\mathcal{F}(\varepsilon)$ spanned by the functions $R(\cdot, t)$ for $t \in T_n$, i.e.,

$$V_{T_n} = \{g : [0,1] \to \mathbb{R}\ \text{ with }\ g(\cdot) = \mathbb{E}(U\varepsilon(\cdot))\ \text{ where }\ U \in L(\varepsilon(t), t \in T_n)\}.$$

Our task is to prove that if $R_{|T_n} = (R(t_i,t_j)_{1\le i,j\le n})$ is a non-singular matrix then V_{T_n} is a closed subspace of $\mathcal{F}(\varepsilon)$. For this let, $(g_m)_{m\ge1}$ be a sequence in V_{T_n} converging to $g \in \mathcal{F}(\varepsilon)$. We shall prove that $g \in V_{T_n}$. Note that,

$$g_m(t) = \mathbb{E}(U_m\varepsilon(t)) \quad \text{with} \quad U_m = \sum_{i=1}^{n} a_{i,m}\varepsilon(t_i), \quad \text{where} \quad (a_{i,m})_{m\ge1} \in \mathbb{R}.$$

Since g_m converges in $\mathcal{F}(\varepsilon)$ then it is a Cauchy sequence, i.e.,

$$\lim_{m_1,m_2\to\infty} ||g_{m_1} - g_{m_2}||^2 = 0.$$

By the definition of the norm on $\mathcal{F}(\varepsilon)$ we have,

$$\begin{aligned} ||g_{m_1} - g_{m_2}||^2 &= \mathbb{E}((U_{m_1} - U_{m_2})^2) = \mathbb{E}\Big(\big(\sum_{i=1}^{n}(a_{i,m_1} - a_{i,m_2})\varepsilon(t_i)\big)^2\Big) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{i,m_1} - a_{i,m_2})(a_{j,m_1} - a_{j,m_2})R(t_i,t_j) = A'_{m_1,m_2}R_{|T_n}A_{m_1,m_2}, \end{aligned}$$

where $A'_{m_1,m_2} = (a_{1,m_1} - a_{1,m_2}, \cdots, a_{n,m_1} - a_{n,m_2})'$. Thus,

$$\lim_{m_1,m_2\to\infty} A'_{m_1,m_2}R_{|T_n}A_{m_1,m_2} = 0.$$

Since $R_{|T_n}$ is a symmetric positive matrix, we obtain,

$$\lim_{m_1,m_2\to\infty} A'_{m_1,m_2} = \lim_{m_1,m_2\to\infty}(a_{1,m_1} - a_{1,m_2}, \cdots, a_{n,m_1} - a_{n,m_2})' = (0,\ldots,0)',$$

which yields that $(a_{i,m})_m$ is a Cauchy sequence on $\mathbb{R}$ for all $i = 1,\cdots,n$. Taking $a_i = \lim_{m\to\infty} a_{i,m}$ we obtain by the uniqueness of the limit,

$$g(\cdot) = \mathbb{E}(U\varepsilon(\cdot)) \ \text{ with } \ U = \sum_{i=1}^{n} a_i\varepsilon(t_i),$$

which yields that $g \in V_{T_n}$. Hence V_{T_n} is closed. $\square$

Since V_{T_n} is a closed subspace in the Hilbert space $\mathcal{F}(\varepsilon)$, one can define the orthogonal projection operator from $\mathcal{F}(\varepsilon)$ to V_{T_n} which we note by $P_{|T_n}$, i.e., for every $f \in \mathcal{F}(\varepsilon)$,

$$P_{|T_n}f = \underset{g\ \in V_{T_n}}{\operatorname{argmin}} ||f - g||.$$

Par definition of $P_{|T_n}$, we have for any $g \in V_{T_n}$

$$\langle P_{|T_n}f - f, g\rangle = 0.$$

Now, for $t_i \in T_n$, $R(\cdot,t_i) \in V_{T_n}$. Hence, for every $i = 1,\ldots,n$.

$$\langle P_{|T_n}f - f, R(\cdot,t_i)\rangle = 0 \ \text{ or equivalently } \ \langle P_{|T_n}f, R(\cdot,t_i)\rangle = \langle f, R(\cdot,t_i)\rangle.$$

The last equality, together with (100), gives that,

$$P_{|T_n}f(\cdot) = f(\cdot) \ \text{ on } T_n. \ \square \tag{104}$$

Supplementary facts

(F1) Let f be defined by (101). We shall prove that if $g \in V_{T_n}$, i.e., if $g(\cdot) = \sum_{j=1}^{n} a_j R(t_j, \cdot)$ for some $a_i \in \mathbb{R}$, then

$$||f - g||^2 = \int_0^1 \varphi(s)(f(s) - g(s))\, ds - \sum_{i=1}^{n} a_i(f(t_i) - g(t_i)).$$

In fact,

$$||f - g||^2 = \langle f - g, f - g\rangle = \langle f, f - g\rangle - \langle g, f - g\rangle$$

On the one hand, note that $f - g \in \mathcal{F}(\varepsilon)$ and by using (100) we obtain,

$$\langle g, f - g\rangle = \sum_{i=1}^{n} a_i \langle R(t_i, \cdot), f - g\rangle = \sum_{i=1}^{n} a_i(f(t_i) - g(t_i)). \tag{105}$$

On the another hand, Lemma 5 and its proof yield that $f(\cdot) = \mathbb{E}(X\varepsilon(\cdot))$ where $X \in L_2(\varepsilon)$ and that,

$$\lim_{l\to\infty} \mathbb{E}(X_l - X)^2 = 0 \quad \text{where} \quad X_l = \sum_{j=1}^{l-1} (x_{j+1,l} - x_{j,l})\varphi_{x,h}(x_{j,l})\varepsilon(x_{j,l}),$$

where $(x_{j,l})_{j=1,\cdots,l}$ is a suitable partition of $[0, 1]$. Let $F_l(\cdot) = \mathbb{E}(X_l\varepsilon(\cdot))$ which is an element of $\mathcal{F}(\varepsilon)$. Clearly,

$$\langle f, f - g\rangle = \langle f - F_l, f - g\rangle + \langle F_l, f - g\rangle.$$

We have,

$$|\langle f - F_l, f - g\rangle| \le ||f - F_l||\, ||f - g|| \le \sqrt{\mathbb{E}((X_l - X)^2)}||f - g||.$$

Thus $\lim_{l\to\infty} \langle f - F_l, f - g\rangle = 0$. In addition,

$$\langle F_l, f - g\rangle = \Big\langle \sum_{j=1}^{l-1} (x_{j+1,l} - x_{j,l})\varphi(x_{j,l})R(x_{j,l}, \cdot), f - g\Big\rangle$$

$$= \sum_{j=1}^{l-1} (x_{j+1,l} - x_{j,l})\varphi(x_{j,l})\langle R(x_{j,l}, \cdot), f - g\rangle = \sum_{j=1}^{l-1} (x_{j+1,l} - x_{j,l})\varphi(x_{j,l})(f(x_{j,l}) - g(x_{j,l})).$$

Hence,

$$\lim_{l\to\infty} \langle F_l, f - g\rangle = \int_0^1 \varphi(t)(f(t) - g(t))\, dt.$$

Finally,

$$\langle f, f - g\rangle = \int_0^1 \varphi(t)(f(t) - g(t))\, dt. \ \Box$$

(F2) For $x \in [0,1]$, let $f_{x,h}$ be defined by (2). We shall prove that,

$$m\mathrm{Var}(\hat{g}_n^{pro}(x)) = ||P_{|T_n} f_{x,h}||^2.$$

In fact, by the definition of the projection operator $P_{|T_n}$, we have $P_{|T_n} f_{x,h} \in V_{T_n}$ and for $t \in [0,1]$,

$$P_{|T_n} f_{x,h}(t) = \sum_{i=1}^{n} a_i R(t_i, t) = \mathbb{E}(\sum_{i=1}^{n} a_i \epsilon(t_i)\epsilon(t)) \quad \text{for some} \quad a_i \in \mathbb{R} \text{ for } i = 1, \cdots, n,$$

and then,

$$||P_{|T_n} f_{x,h}||^2 = \mathbb{E}\left(\sum_{i=1}^{n} a_i \epsilon(t_i)\right)^2 = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j R(t_i, t_j) = \sum_{i=1}^{n} a_i P_{|T_n} f_{x,h}(t_i).$$

Recall that $m_{x,h}{}'_{|T_n} = f_{x,h}{}'_{|T_n} R_{|T_n}$ and using (104) we obtain,

$$P_{|T_n} f_{x,h}(t_i) = f_{x,h}(t_i) = \sum_{j=1}^{n} m_{x,h}(t_j) R(t_i, t_j). \tag{106}$$

We have then, using (106),

$$\begin{aligned} ||P_{|T_n} f_{x,h}||^2 &= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} m_{x,h}(t_j) R(t_i, t_j) = \sum_{j=1}^{n} m_{x,h}(t_j) \sum_{i=1}^{n} a_i R(t_i, t_j) \\ &= \sum_{j=1}^{n} m_{x,h}(t_j) \sum_{i=1}^{n} m_{x,h}(t_i) R(t_i, t_j) = m\mathrm{Var}(\hat{g}_n^{pro}(x)). \ \square \end{aligned}$$

(F3) We shall now prove prove that every function in $\mathcal{F}(\varepsilon)$ is continuous on $[0,1]$. In fact let $g \in \mathcal{F}(\varepsilon)$, i.e.,

$$g(\cdot) = \mathbb{E}(U\varepsilon(\cdot)) \quad \text{for some} \quad U \in L_2(\varepsilon).$$

For $s, t \in [0,1]$, (100) and Cauchy-Swartz inequality yields,

$$\begin{aligned} |g(t) - g(s)| &= |\langle R(\cdot, t), g\rangle - \langle R(\cdot, s), g\rangle| = |\langle R(\cdot, t) - R(\cdot, s), g\rangle| \\ &\le ||R(\cdot, t) - R(\cdot, s)||\ ||g|| = ||R(\cdot, t) - R(\cdot, s)||\ \sqrt{\mathbb{E}(U^2)}. \end{aligned}$$

Since ε is of second order process then $\mathbb{E}(U^2) < \infty$ and since R is continuous on $[0,1]^2$ we obtain,

$$\lim_{s \to t} ||R(\cdot, t) - R(\cdot, s)||^2 = \lim_{s \to t} (R(t,t) + R(s,s) - 2R(s,t)) = 0,$$

which yields that $\lim_{s \to t} |g(t) - g(s)| = 0$. Hence g is continuous. $\square$

(F4) Suppose that R verifies Assumptions $(A), (B)$ and (C). Let f be defined by (101). We shall prove that if $g \in V_{T_n}$, i.e., $g(\cdot) = \sum_{j=1}^{n} a_j R(t_j, \cdot)$ with $(a_i)_i \in \mathbb{R}$ then,

$$f''(t) - g''(t^+) = -\alpha(t)\varphi(t) + \langle R^{(0,2)}(\cdot, t^+), f - g\rangle.$$

In fact, we have, as in Equation (33),

$$f''(t) = -\alpha(t)\varphi(t) + \int_0^1 R^{(0,2)}(s,t^+)\varphi(s)\, ds.$$

In addition, we have clearly

$$g''(t^+) = \sum_{j=1}^{n} a_j R^{(0,2)}(t_j, t^+).$$

Thus,

$$f''(t) - g''(t^+) = -\alpha(t)\varphi(t) + \int_0^1 R^{(0,2)}(s,t^+)\varphi(s)\, ds - \sum_{j=1}^{n} a_j R^{(0,2)}(t_j, t^+).$$

We have,

$$\langle R^{(0,2)}(\cdot,t^+), f-g\rangle = \langle R^{(0,2)}(\cdot,t^+), f\rangle - \langle R^{(0,2)}(\cdot,t^+), g\rangle$$

On the one hand, since by Assumption (C), $R^{(0,2)}(\cdot,t^+)$ is in $\mathcal{F}(\varepsilon)$ then (100) yields,

$$\langle R^{(0,2)}(\cdot,t^+), g\rangle = \sum_{j=1}^{n} a_j \langle R^{(0,2)}(\cdot,t^+), R(\cdot,t_j)\rangle = \sum_{j=1}^{n} a_j R^{(0,2)}(t_j, t^+). \tag{107}$$

On the other hand, from Lemma 5 we have $f(\cdot) = \mathbb{E}(X\varepsilon(\cdot))$ where $X \in L_2(\varepsilon)$ and,

$$\lim_{l\to\infty} \mathbb{E}(X_l - X)^2 = 0 \quad \text{with} \quad X_l = \sum_{j=1}^{l-1} (x_{j+1,l} - x_{j,l})\varphi(x_{j,l})\varepsilon(x_{j,l}),$$

where $(x_{j,l})_{j=1,\cdots,l}$ is a suitable partition of $[0,1]$. Let $F_l(\cdot) = \mathbb{E}(X_l\varepsilon(\cdot)) \in \mathcal{F}(\varepsilon)$, we have,

$$\langle R^{(0,2)}(\cdot,t^+), f\rangle = \langle R^{(0,2)}(\cdot,t^+), f - F_l\rangle + \langle R^{(0,2)}(\cdot,t^+), F_l\rangle, \tag{108}$$

and,

$$|\langle R^{(0,2)}(\cdot,t^+), f - F_l\rangle| \le ||R^{(0,2)}(\cdot,t^+)||\; ||f - F_l|| = ||R^{(0,2)}(\cdot,t^+)|| \sqrt{\mathbb{E}((X_l - X)^2)}.$$

The last bound together with Assumption (C) gives $\lim_{l\to\infty} |\langle R^{(0,2)}(\cdot,t^+), f - F_l\rangle| = 0$, in addition,

$$\begin{aligned}\langle R^{(0,2)}(\cdot,t^+), F_l\rangle &= \sum_{j=1}^{l-1} (x_{j+1,l} - x_{j,l})\varphi(x_{j,l})\langle R^{(0,2)}(\cdot,t^+), \varepsilon(x_{j,l})\rangle \\ &= \sum_{j=1}^{l-1} (x_{j+1,l} - x_{j,l})\varphi(x_{j,l}) R^{(0,2)}(x_{j,l}, t^+).\end{aligned}$$

Thus,

$$\lim_{l\to\infty} \langle R^{(0,2)}(\cdot,t^+), F_l\rangle = \int_0^1 \varphi(s) R^{(0,2)}(s,t^+)\, ds. \tag{109}$$

Finally, using (107), (108) and (109) yield,

$$\langle R^{(0,2)}(\cdot,t^+), f-g\rangle = \int_0^1 \varphi(s) R^{(0,2)}(s,t^+)\, ds - \sum_{j=1}^{n} a_j R^{(0,2)}(t_j, t^+). \;\square$$

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